10-4 Solution of the Vlasov Equation

In Secs. 8-2 and 10-2 we obtained solutions to the Landau and cyclotron damping problems by calculating the perturbations to particle trajectories due to their passage through a first-order electric field. A related method has been used for the solution of the collisionless Boltzmann equation by J. E. Drummond (1958), R. Z. Sagdeev and V. D. Shafranov (1958), and also by M. N. Rosenbluth and N. Rostoker (1958). In this method the first-order perturbation to the velocity distribution function is calculated in the Lagrangian system of coordinates, that is, in coordinates that follow the zero-order trajectory of the particles. Knowledge of the perturbed velocity distribution in terms of the first-order electric field allows one, by taking moments, to calculate the macroscopic charge and current density and also the susceptibility tensors. Substitution into Maxwell's equation then gives the dispersion relation as the condition for nontrivial solutions in the absence of sources.

To begin, we note that if a trajectory is defined by

$$r = r(t),$$

then the rate of change of the distribution function $f$ as one moves along the trajectory is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} \frac{dr}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt},$$

(25)

where $f = f(r,p,t)$, $p = mv$, and $m = m_0 (1 - v^2/c^2)^{-1/2}$ is the relativistic mass. The extension to relativistic velocities in the calculation of susceptibilities causes almost no additional complication, and we retain the relativistic formalism through much of this chapter.

The zero-order trajectory of a charged particle of type $s$ in a static magnetic field is given by

$$\frac{dt}{dt} = v \quad \text{and} \quad \frac{dp}{dt} = q_s c \times B_0.$$  

(26)

* $\partial/\partial p$ denotes $\hat{\mathbf{x}} \partial/\partial p_x + \hat{\mathbf{y}} \partial/\partial p_y + \hat{\mathbf{z}} \partial/\partial p_z$. 

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Therefore, the rate of change of $f_s$ along the zero-order trajectory is:

$$\left( \frac{df_s}{dt} \right)_0 = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{\mathbf{v}}{c} \times \mathbf{B}_0 \cdot \frac{\partial f_s}{\partial \mathbf{p}}.$$  \hspace{1cm} (27)

Looking now at the Vlasov equation, Eq. (8-26), we see that the zero-order kinetic equation (\(E = 0, B = B_0\)) is given by

$$\left( \frac{df_{s0}}{dt} \right)_0 = 0.$$  \hspace{1cm} (28)

The first-order Vlasov equation may be written

$$\left( \frac{df_{s1}}{dt} \right)_0 = -q_s \left( \mathbf{E}_1 + \frac{\mathbf{v}}{c} \times \mathbf{B}_1 \right) \cdot \frac{\partial f_{s0}}{\partial \mathbf{p}}.$$  \hspace{1cm} (30)

in which the left-hand side is the rate of change of $f_{s1}$ along the zero-order trajectory in \((\mathbf{r}, \mathbf{p}, t)\) space. The problem is then solved by carrying out the integration

$$f_{s1}(\mathbf{r}, \mathbf{v}, t) = -q_s \int_{-\infty}^{t} dt' \left[ \mathbf{E}_1(\mathbf{r}', t') + \frac{\mathbf{v}}{c} \times \mathbf{B}_1(\mathbf{r}', t') \right] \cdot \frac{\partial f_{s0}(\mathbf{p}')}{\partial \mathbf{p}'}.$$  \hspace{1cm} (31)

along this trajectory from time $t' = -\infty$ to $t' = t$. Rigorously, the integration should have been written in a form similar to Eq. (8-31),

$$f_{s1}(\mathbf{r}, \mathbf{p}, t) = -q_s \int_{t_0}^{t} dt' \mathcal{G}_{s1}[\mathbf{r}_{gc}(\mathbf{r}, \mathbf{p}, t), \mathbf{p}_s, \mathbf{p}_\parallel],$$  \hspace{1cm} (32)

where the arguments of $g_{s1}$ are constants of the zero-order motion. These constants include $\mathbf{p}_s, \mathbf{p}_\parallel$ and the position of the guiding center for the spiralling trajectory, $\mathbf{r}_{gc}(\mathbf{r}, \mathbf{p}, t)$. $\mathbf{r}_{gc}$ moves parallel to $B_0$ with velocity $v_\parallel = p_\parallel / m$, but its projection perpendicular to $B_0$ does not change (Prob. 2).

Mathematically, $g_{s1}(\mathbf{r}_{gc}, \mathbf{p}, p_\parallel)$ in Eq. (32) is a solution to the homogeneous differential equation, i.e., Eq. (30) with the right-hand side replaced by zero. And the integral in Eq. (32) is a particular solution, that is, a solution
to Eq. (32) for the specific inhomogeneous term on its right-hand side. The total solution for $f_{31}(r,p,t)$ is the sum of the two, with the homogeneous solution chosen to fit the initial conditions, $f_{31}(r_0,p_0,t_0) = g_{31}(r_0,p_0,\varphi_0)$. At this stage $E_1(r,t)$ and $B_1(r,t)$ may be considered known quantities and we have not yet claimed self-consistency. That is, we have not yet demanded that $E_1$ and $B_1$ are induced by the plasma space charge and currents stemming from $f_{31}(r,p,t)$. Rather, we view Eq. (32) as offering the response, $f_{31}(r,p,t)$, to a known electromagnetic field.

In physical terms, $g_{31}$ in Eq. (32) is the “ballistic” contribution to $f_{31}(r,p,t)$ and would correspond to the evolution of the initial conditions in the absence of any first-order electromagnetic field. The amplitude of the moments of $g_{31}$ may decay in time due to phase mixing (Prob. 8-11), but there is no reason for $g_{31}$ to grow. On the other hand, let us now consider the case that $E_1(r,t)$ and $B_1(r,t)$ grow exponentially in time. Although $f_{31} = g_{31}$ at the historical time $t = t_0$ and hence $g_{31}$ has affected the structure of $f_{31}$ for all later times, the relation between $f_{31}$, $E_1$ and $B_1$, as they each grow larger and larger, will asymptotically evolve independently of the current value of the nongrowing $g_{31}$. It is this asymptotic relation that is represented by Eq. (31). Expressions of the corresponding relationship between $f_{31}$, $E_1$, and $B_1$ for steady-state and damped solutions may then be obtained from Eq. (31) by analytic continuation.
An alternative interpretation of Eq. (31) is possible, using a model that bears considerable resemblance to physical reality. Let us consider a plasma with collisions in which the asymptotic electromagnetic field already has been set up and assume, as in Sec. 8-5, that the collisions tend to remove position-velocity correlations and to restore the distribution function to its zero-order form. In time any initial perturbation of the distribution function will be forgotten and particles will be accelerated from random initial positions and velocities by the asymptotic electric field. It is this evolution which, in the limit of a slow collision rate, is also described by Eq. (31).

One point is especially significant here. If we consider, for simplicity, an oscillating electric field of constant amplitude, the forced oscillations of the nonresonant particles will be the same, on the average, no matter whether they were randomized recently or long ago. However, the resonant particles are steadily accelerated, and their coherent velocity depends on the elapsed time since randomization occurred. Nevertheless, the dynamics of the damping process happen to work out that fewer and fewer resonant particles remain in phase and are accelerated to higher and higher first-order velocities in just such a proportion that the first-order macroscopic velocity retains constant magnitude (Prob. 8-4). Now the electromagnetic field in the plasma is induced by the macroscopic charge and current densities, and it is this self-consistent electromagnetic field that accelerates the individual charged particles. Because of the above-described idiosyncrasy of the macroscopic behavior, the asymptotic electromagnetic field in a plasma will be independent of elapsed time since randomization. In other words, the same electromagnetic field will appear whether plasma particles undergo rapid collisions, and many particles are placed in phase with the field but remain in phase for only a short time until another collision knocks them out of phase, or whether collisions are infrequent and only a few particles remain in phase but for a long time. In particular, since the electromagnetic field and the charge and current density are independent of the randomization rate, the power that goes into Landau and cyclotron damping is also independent of this rate.

A second consequence of this macroscopic idiosyncrasy is a justification for the use of the asymptotic electromagnetic field in the orbit integration, Eq. (31). We now know that we are entitled to let collisions become arbitrarily infrequent, at least within the framework of the linear theory, so that the calculation may be applied to the collisionless plasma. More important, however, is the inverse consideration. We know that the model of the collisionless plasma runs into difficulty when trajectories start to deviate appreciably from the trajectories predicted from linear theory. Therefore we can introduce a collision rate that is slow enough so that the macroscopic quantities very nearly reach their asymptotic values [see Eq. (9)] but fast enough so that the linearized theory still applies [Eqs. (19) and (20)]. Repeating the conclusion reached in Sec. 8-6, it is this range of collision rates that produces an environment in which the linearized theory is valid.

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After this rather extended justification of methods, we finally set ourselves to the actual task of calculating $f_{s1}$ and its lowest moments. The remainder of the chapter is devoted to the details of this calculation.

10-5 Transformation from Lagrangian to Eulerian Coordinates

In this section we shall be dealing with distribution functions for particles that all have the same charge-to-mass ratio (e.g., with electrons, or protons, or deuterons). We shall, consequently, drop the subscript $s$. We now turn to the evaluation of the integral in Eq. (31) for particles of a single charge-to-mass ratio and with a zero-order distribution function $f_0(p)$. We substitute the asymptotic field $E_1(r',t') = E \exp(\xi k \cdot r' - i\omega t')$ and use Maxwell's induction equation, $B_1 = (k \omega/c) \times E_1$, to replace $B_1$. Equation (31) becomes

$$f_1(r,p,t) = \frac{-q}{\pi \omega} \int_{-\infty}^{t} dt' e^{i k \cdot r - i \omega t'}$$

$$\times E \cdot \left[ 1 \left( 1 - \frac{v' \cdot k}{\omega} \right) + \frac{v' k}{\omega} \right] \frac{\partial f_0(p')}{\partial p'},$$

(33)

where $1$ is the unit dyadic.

The integrand is to be evaluated in the Lagrangian system of coordinates, that is, along the zero-order trajectory, $r'(t')$. The end point of the trajectory is $r' = r$ at $t' = t$. For earlier times, the trajectory obeys the zero-order equation of motion, Eq. (26),

$$\frac{dp'}{dt} = \frac{qB_0}{c} v' \times \hat{z} = \Omega p' \times \hat{z},$$

(34)

where

$$p = m \frac{dr}{dt} = mv = \gamma m_0 v, \quad E = mc^2,$$

$$\gamma^2 = \left( 1 - \frac{v^2}{c^2} \right)^{-1} = \frac{(p)^2}{m_0^2c^2} + 1,$$

(35)

$$\Omega = \frac{qB_0}{mc} = \frac{\Omega_0}{\gamma}, \quad \Omega_0 = \frac{qB_0}{m_0c}.$$
\( \Omega \) and \( \Omega_0 \) are algebraic quantities. Because the acceleration is always perpendicular to \( \mathbf{p} \), the particle energy, together with \( \Omega \), \( v \), and \( \gamma \), remain constant. Then the solution of Eq. (34) that reaches \( \mathbf{r}' = \mathbf{r} \) and \( \mathbf{v}' = \mathbf{v} \) at \( t' = t \) may be conveniently expressed in terms of the Eulerian coordinates \( \mathbf{r} \), \( \mathbf{v} \), \( t \) by the set, Prob. 2,

\[
\tau = t - t',
\]

\[
v_x = v_1 \cos \phi, \quad v_y = v_1 \sin \phi,
\]

\[
v'_x = v_1 \cos(\phi + \Omega \tau), \quad v'_y = v_1 \sin(\phi + \Omega \tau),
\]

\[
v'_z = v_{||},
\]

\[
x' = x - \frac{v_1}{\Omega} [\sin(\phi + \Omega \tau) - \sin \phi],
\]

\[
y' = y + \frac{v_1}{\Omega} [\cos(\phi + \Omega \tau) - \cos \phi],
\]

\[
z' = z - v_{||} \tau.
\]

Next, for substitution into Eq. (33), we write

\[
k_x = k_1 \cos \theta, \quad k_y = k_1 \sin \theta,
\]

\[
\mathbf{k} \cdot \mathbf{r}' - \omega t' = \mathbf{k} \cdot \mathbf{r} - \omega t + \beta,
\]

(37)
\[ \beta = - \frac{k_1 v_1}{\Omega} \left[ \sin(\phi - \theta + \Omega \tau) - \sin(\phi - \theta) \right] + (\omega - k_v v_\parallel) \tau. \]

Then we define

\[ U = \frac{\partial f_0}{\partial p_1} + \frac{k_v}{\omega} \left( v_\parallel \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_1} \right), \]

\[ V = \frac{k_1}{\omega} \left( v_\parallel \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_1} \right), \]

\[ W = \left( 1 - \frac{n \Omega}{\omega} \right) \frac{\partial f_0}{\partial p_\parallel} + \frac{n \Omega p_\parallel}{\omega p_1} \frac{\partial f_0}{\partial p_1}. \]

The quantity \( W \), defined here, appears only in Eqs. (45), (46), and (56). Finally, substituting from Eqs. (36)-(38), Eq. (33) may be written, Prob. 3,

\[ f_1(r,p,t) = - q e^2 \cdot \tau - io \int_0^\infty d\tau e^{i\beta} \left[ E_x U \cos(\phi + \Omega \tau) \right. \]

\[ + E_y U \sin(\phi + \Omega \tau) + \left. E_z \left( \frac{\partial f_0}{\partial p_\parallel} - V \cos(\phi - \theta + \Omega \tau) \right) \right]. \]

An alternative method of solution of the first-order Vlasov equation, using guiding-center coordinates, is initiated in Probs. 12-7 and 12-8 and discussed in detail in L. Chen (1987).
10-6 Susceptibilities for Arbitrary $f_0(p_1, p_\parallel)$

It was discussed in Sec. 1-2 that the dielectric tensor is additive in its components. Not only do electrons and the various ion species contribute separately identifiable components to $\epsilon(\omega, k)$, as in Eqs. (1-19)-(1-22), but the contributions to $\epsilon$ from different portions of a velocity distribution (e.g., from the cyclotron-resonant ions) may also be identified and tracked. The governing relation is simply Eq. (1-4),

$$\epsilon(\omega, k) = 1 + \sum_s \chi_s(\omega, k). \quad (40)$$

Having obtained $f_{s1}(r, p, t)$ in Eq. (39) for a hot magnetized plasma, we are now able to take the velocity moments to find the contributions to the order plasma current $j(\omega, k)$ for each species, and thus evaluate their susceptibilities, $\chi_s(\omega, k)$. Using Eq. (1-5),

$$j = \sum_s j_s = \sum_s q_s \int d^3p \nabla f_{s1}(r, p, t) = -\frac{i\omega}{4\pi} \sum_s \chi_s \cdot E. \quad (41)$$

As a modest simplification, we again set $k_\parallel = 0$, that is, $\theta = 0$ [but see Eq. (64) below]. It was pointed out by D. C. Montgomery and D. A. Tidman (1965) that the identities

$$e^{iz \sin \phi} = \sum_{n = -\infty}^{\infty} e^{in\phi} J_n(z),$$

$$e^{-iz \sin(\phi + \Omega r)} = \sum_{m = -\infty}^{\infty} e^{-im(\phi + \Omega r)} J_m(z), \quad (42)$$
together with their derivatives with respect to $\phi$ and $z$, lead by orthogonality quickly to

$$\int_{0}^{2\pi} d\phi e^{-iz [\sin(\phi + \Omega \tau) - \sin \phi]}$$

$$\begin{pmatrix}
\sin \phi \sin(\phi + \Omega \tau) \\
\sin \phi \cos(\phi + \Omega \tau) \\
\cos \phi \sin(\phi + \Omega \tau) \\
\cos \phi \cos(\phi + \Omega \tau) \\
1 \\
\sin \phi \\
\cos \phi \\
\sin(\phi + \Omega \tau) \\
\cos(\phi + \Omega \tau)
\end{pmatrix}$$

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$$2\pi \sum_{n = -\infty}^{\infty} e^{-i n \Omega \tau}$$

$$\begin{pmatrix}
(J_n')^2 \\
- \frac{in}{z} J_n J_n' \\
\frac{in}{z} J_n J_n'' \\
\frac{n^2}{z} J_n^2 \\
\frac{n}{z} J_n \\
i J_n J_n' \\
\frac{n}{z} J_n^2 \\
i J_n J_n'' \\
\frac{n}{z} J_n \\
i J_n J_n'' \\
\frac{n}{z} J_n^2
\end{pmatrix}$$

$$ \text{(43)} $$
Additional entries for this set will be found in Table 14-1. In set (43), \( J_n \) denotes \( J_n(z) \), \( z \) denotes \( k_z v_z / \Omega \), and \( \Omega = \Omega(\mathbf{p}_1, p_\parallel) \) is the relativistic cyclotron frequency, Eq. (35). The set (43) permits immediate integration of \( \phi \), the gyrophase angle, over these moments of \( f_1 \) in Eq. (39). Using Eq. (37) with \( \theta = 0 \), as assumed above, and Eq. (43), the integrations over \( \tau \) are all of the form

\[
- q \int_0^\infty d\tau \exp\left[i(\omega - k_\parallel v_\parallel - n\Omega)\tau\right] = \frac{-iq}{\omega - k_\parallel v_\parallel - n\Omega},
\]

provided \( \omega > 0 \). Knowledge of the velocity moments now leads immediately to the species contributions to first-order plasma current and thus to the susceptibility tensors, Eq. (41). Normalizing \( \int d^3p f_0^{(s)}(p) = 1 \), Prob. 4,

\[
\chi_s = \frac{\omega p_{\parallel,3}}{\omega \Omega_{0,3}} \sum_{n=-\infty}^{\infty} \int_0^\infty 2\pi p_\parallel \, dp_\parallel \int_{-\infty}^{\infty} \, dp_\perp \left( \frac{\Omega}{\omega - k_\parallel v_\parallel - n\Omega} S_n \right),
\]

\[
S_n = \begin{bmatrix}
\frac{n^2 J_n^2}{z} p_\parallel U & -i n J_n J'_n p_\parallel U & i n J_n J'_n p_1 W \\
-i n J_n J'_n p_\parallel U & \frac{(J'_n)^2}{z} p_\parallel U & -i n J_n J'_n p_1 W \\
\frac{n^2 J_n^2}{z} p_\parallel U & i n J_n J'_n p_\parallel U & J_n^2 p_\parallel W
\end{bmatrix},
\]

10-6 Susceptibilities for Arbitrary \( f_0(p_1, p_\parallel) \)

where again \( J_n = J_n(z) \) and \( z = k_z v_z / \Omega \). The use of the Landau contour for the integral over \( p_\parallel \) will extend the validity of this expression to cases where \( \text{Im} \omega < 0 \). \( \omega_{0,s} \) and \( \Omega_{0,s} \) are the nonrelativistic plasma and cyclotron frequencies for particles of species \( s \). A few steps will verify that

\[
p_\parallel U - p_1 W = \left(p_\parallel \frac{\partial f}{\partial p_\parallel} - p_1 \frac{\partial f}{\partial p_\parallel}\right) \frac{\omega - k_\parallel v_\parallel - n\Omega}{\omega}.
\]
Noting then that
\[
\sum_{n = -\infty}^{\infty} n J_n^2 = 0, \quad \sum_{n = -\infty}^{\infty} J_n J'_n = 0, \quad \sum_{n = -\infty}^{\infty} J_n^2 = 1, \quad (47)
\]
it is clear that the factor $p_1 W$ in $S_{xz}$ and in $S_{yz}$ in Eq. (45) can be replaced by $p_1 U$. The coefficients of $\chi_s$ therefore display the symmetry of the Onsager relations even for an arbitrary $f_0(p_1, p_\parallel)$. The identities just cited allow a rearrangement also of $\chi_{xz}$, so that finally
\[
\chi_s = \frac{\omega^2 \rho_{0z}}{\omega \Omega_{0z}} \int_0^\infty 2\pi p_1 \ dp_1 \int_{-\infty}^{\infty} dp_\parallel \left[ \epsilon_\parallel \epsilon_\parallel \frac{\Omega}{\omega} \left( \frac{1}{p_\parallel} \frac{\partial f_0}{\partial p_\parallel} - \frac{1}{p_1} \frac{\partial f_0}{\partial p_1} \right) p_\parallel^2 \right.
\]
\[
+ \sum_{n = -\infty}^{\infty} \frac{\Omega p_1 U}{\omega - k_\parallel v_\parallel - n\Omega} T_n. \quad (48)
\]

\[
T_n = \begin{pmatrix}
\frac{n^2 J_n^2}{z^2} & \frac{inJ_n J'_n}{z} & \frac{nJ_n^2}{zp_1} \\
- \frac{inJ_n J'_n}{z} & (J'_n)^2 & \frac{-iJ_n J'_n p_\parallel}{p_1} \\
\frac{nJ_n^2}{zp_1} & \frac{iJ_n J'_n p_\parallel}{p_1} & \frac{J_n^2 p_\parallel^2}{p_1^2}
\end{pmatrix}
\]

The Landau contour is to be used for the integral over $p_\parallel$ and again the argument of the Bessel functions is $z = k_\parallel v_\parallel / \Omega$, and $\Omega = \Omega_s(p_1, p_\parallel)$ is the relativistic cyclotron frequency, Eq. (35).

Other forms for the susceptibilities may also be derived. For instance, following K. Miyamoto (1980) in using the additional identities

*Based on broad arguments from the theory of thermodynamics of irreversible processes, it may be shown in general that transport coefficients relating fluxes and forces show a symmetry $T_{ij}(B_0) = T_{ji}(-B_0)$. In Eq. (48), $\chi(-B_0)$ may be computed from $\chi(B_0)$ simply by replacing $\Omega$ by $-\Omega$. For further discussion of the Onsager relations, see, for example, S. R. deGroot and P. Mazur (1962) and R. Balescu (1991).
\[
\frac{U}{\omega - k_{\parallel}v_{\parallel} - n\Omega} = \frac{1}{\omega} \frac{\partial f_0}{\partial p_1} + \frac{(k_{\parallel}v_{\parallel}/\omega)(\partial f_0/\partial p_{\parallel}) + (n\Omega/\omega)(\partial f_0/\partial p_1)}{\omega - k_{\parallel}v_{\parallel} - n\Omega},
\]

\[
\sum_{n = -\infty}^{\infty} (J'_n)^2 = \frac{1}{2}, \quad \sum_{n = -\infty}^{\infty} \frac{n^2 J_n^2(z)}{2} = \frac{1}{2}, \quad \sum_{n = -\infty}^{\infty} nJ_n J'_n = 0,
\]

one may easily obtain

\[
\chi_s = \frac{\omega^2 \rho_{0,s}}{\omega \Omega_{0,s}} \int_0^{\infty} 2\pi p_1 dp_1 \int_{-\infty}^{\infty} dp_{\parallel} \left[ \frac{\Omega}{2\omega} p_1 \frac{\partial f_0}{\partial p_1} + \frac{\Omega}{\omega} p_{\parallel} \frac{\partial f_0}{\partial p_{\parallel}} \right.
\]
\[
+ \frac{\Omega}{2\omega} \left( 2p_{\parallel} \frac{\partial f_0}{\partial p_{\parallel}} - p_1 \frac{\partial f_0}{\partial p_1} \right)
\]
\[
+ \sum_{n = -\infty}^{\infty} \frac{\Omega p_1}{\omega - k_{\parallel}v_{\parallel} - n\Omega} \left( \frac{k_{\parallel}v_{\parallel}}{\omega} \frac{\partial f_0}{\partial p_{\parallel}} + \frac{n\Omega}{\omega} \frac{\partial f_0}{\partial p_1} \right) T_n \bigg],
\]

(50)
Several different circumstances can lead to the simplification of these expressions. For example, isotropy in \( f_0(p_\perp, p_\parallel) = f_0(p_\perp^2 + p_\parallel^2) \) will simplify the form of \( U \) and \( W \) in Eq. (38), and will reduce \( V \) to zero. Restriction to nonrelativistic energies will eliminate the dependence of \( \Omega \),

\[
\Omega = \frac{\Omega_0}{\left[1 + (p_\perp^2 + p_\parallel^2)/m_0^2c^2\right]^{1/2}},
\]

on \( p_\perp \) and \( p_\parallel \). And when \( \Omega \) is no longer a function of \( p_\perp \) and \( p_\parallel \), those integrals in Eqs. (48) and (50) that do not have a resonant denominator may be performed trivially.

Finally, the specification of a nonrelativistic Maxwellian distribution for perpendicular velocities, applicable to most cases considered (a loss-cone distribution for a plasma confined by magnetic mirrors is an important exception, but see Prob. 10) allows us to carry out the integral over \( p_\perp \). The next section addresses this calculation.

10-7 Susceptibilities for a Maxwellian \( f_0(v_\perp) \)

10-7 Susceptibilities for a Maxwellian \( f_0(v_\perp) \)

In many cases of interest for magnetized plasmas, the velocity distribution for motions perpendicular to \( B_0 \) may be characterized as a nonrelativistic Maxwellian or as a linear combination of Maxwellsians (see Prob. 10, for example). Still retaining the option that the parallel distribution may be non-Maxwellian, albeit also nonrelativistic, we write

\[
f_0(v_\perp, v_\parallel) = h(v_\parallel) \frac{1}{\pi w_\perp^2} \exp\left(-\frac{v_\perp^2}{w_\perp^2}\right),
\]

\[
v_\perp^2 = v_x^2 + v_y^2, \quad w_\perp^2 = \frac{2kT_i}{m},
\]

\[
\int_{-\infty}^{\infty} dv_\parallel h(v_\parallel) = 1.
\]

Note that the normalization is now \( \int d^3v f_0(v_\perp, v_\parallel) = 1 \), whereas in the previous section we used \( \int d^3p f_0(p_\perp, p_\parallel) = 1 \).
The evaluation of susceptibilities for a magnetized nonrelativistic plasma, MaxweIlian in $f_0(u_1)$, is facilitated by an unbelievably apt identity, G. N. Watson (1922),

$$\int_0^\infty \frac{dt}{J_\nu(at)J_\nu(bt)} e^{-\rho^2} = \frac{1}{2\rho^2} \exp\left(-\frac{a^2 + b^2}{4\rho^2}\right) I_\nu\left(\frac{ab}{2\rho^2}\right),$$  

(53)

valid for $\Re \nu > -1$ and $|\arg \rho| < \pi/4$. From Eq. (53) together with its derivatives with respect to $a$ and/or $b$, it is easily found that

$$\frac{1}{\pi \nu_1^{\frac{1}{2}}} \int_0^\infty 2\pi v_1 d\nu_1 J_n\left(\frac{k_1 v_1}{\Omega}\right) e^{-v_1^2/\nu_1^2} = e^{-\lambda I_n(\lambda)},$$

$$\frac{1}{\pi \nu_1^{\frac{1}{2}}} \int_0^\infty 2\pi v_1^2 d\nu_1 J_n\left(\frac{k_1 v_1}{\Omega}\right) J_n\left(\frac{k_1 v_1}{\Omega}\right) e^{-v_1^2/\nu_1^2}$$

$$= -\frac{k_1 \omega_1^{\frac{1}{2}}}{2\Omega} e^{-\lambda [I_n(\lambda) - I_n'(\lambda)]},$$

(54)

$$\frac{1}{\pi \nu_1^{\frac{1}{2}}} \int_0^\infty 2\pi v_1 d\nu_1 \left[J_n\left(\frac{k_1 v_1}{\Omega}\right)\right]^2 e^{-v_1^2/\nu_1^2}$$

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$$\frac{\omega_1^2}{2} e^{-\lambda \left[\frac{n^2}{\lambda} I_n(\lambda) + 2\lambda I_n - 2\lambda I_n'\right]},$$

where

$$\lambda = \frac{k_1^2 \omega_1^{\frac{1}{2}}}{2\Omega} = \frac{1}{2} k_1^2 \langle \rho^2 \rangle = \frac{1}{2} k_1^2 \langle v_x^2 + v_y^2 \rangle \Omega^2 = \frac{k_1^2 \kappa T_\perp}{m\Omega^2}. $$  

(55)

Noting now from Eq. (38) that, in the nonrelativistic regime,

$$p_1 U = v_1 \frac{\partial f_0}{\partial v_1} + \frac{k_1 v_1}{\omega} \left(v_1 \frac{\partial f_0}{\partial v_\parallel} - v_\parallel \frac{\partial f_0}{\partial v_1}\right),$$

$$p_\parallel W = v_\parallel \frac{\partial f_0}{\partial v_\parallel} \left(1 - \frac{n\Omega}{\omega}\right) + \frac{n\Omega v_\parallel^2}{\omega v_1} \frac{\partial f_0}{\partial v_1},$$

(56)
one may evaluate the susceptibility tensor [Eq. (45), (48), or (50)] to find, Prob. 5.

\[
\chi_s = \left[ \begin{array}{c} \hat{e}_\parallel \hat{e}_\parallel \frac{2 \omega_p^2}{\omega k_{||} \omega_{\parallel}^3} \langle v_{\parallel} \rangle + \frac{\omega_p^2}{\omega} \sum_{n=\infty}^{\infty} e^{-\lambda n} Y_n(\lambda) \end{array} \right], \tag{57}
\]

\[
Y_n(\lambda) = \left( \begin{array}{ccc}
\frac{n^2 I_n}{\lambda} A_n & - i(n - I_n') A_n & \frac{k_1 n I_s}{\Omega} B_n \\
\frac{n^2 I_n}{\lambda} & - i(n - I_n') A_n & \frac{k_1 n I_s}{\Omega} B_n \\
\frac{k_1 n I_n}{\Omega} B_n & - i(k_1 - I_n') B_n & \frac{2(\omega - n\Omega)}{k_{||} \omega_{\parallel}^2} I_n B_n \\
\end{array} \right)
\]

\( I_n = I_n(\lambda) \) is the modified Bessel function with argument \( \lambda \), Eq. (55),

\[
I_n(\lambda) = i^{-n} J_n(i\lambda) \tag{58}
\]

\[
= \frac{1}{n!} \left( \frac{\lambda}{2} \right)^n \left[ 1 + \frac{(\lambda/2)^2}{1(n + 1)} + \frac{(\lambda/2)^4}{1 \cdot 2(n + 1)(n + 2)} + \frac{(\lambda/2)^6}{1 \cdot 2 \cdot 3(n + 1)(n + 2)(n + 3)} + \cdots \right],
\]

\( I_n = I_n(\lambda) \) is the modified Bessel function with argument \( \lambda \), Eq. (55),

\[
I_n(\lambda) = i^{-n} J_n(i\lambda) \tag{58}
\]

\[
= \frac{1}{n!} \left( \frac{\lambda}{2} \right)^n \left[ 1 + \frac{(\lambda/2)^2}{1(n + 1)} + \frac{(\lambda/2)^4}{1 \cdot 2(n + 1)(n + 2)} + \frac{(\lambda/2)^6}{1 \cdot 2 \cdot 3(n + 1)(n + 2)(n + 3)} + \cdots \right],
\]
and $I_n = (d/d\lambda) I_n(\lambda)$. The equations in (58) are valid when $n$ is a positive integer or zero. When $n$ is a negative integer, one may use $I_n(\lambda) = I_{-n}(\lambda)$. The asymptotic expansion for $e^{-\lambda} I_n(\lambda)$ is given in Eq. (11-91) and a pertinent summation formula appears in Eq. (11-86). In Eq. (57), $A_n$ and $B_n$ are defined in terms of $H(v_\parallel)$,

$$A_n = \int_{-\infty}^{\infty} dv_\parallel \frac{H(v_\parallel)}{\omega - k_\parallel v_\parallel - n\Omega},$$

$$B_n = \int_{-\infty}^{\infty} dv_\parallel \frac{v_\parallel H(v_\parallel)}{\omega - k_\parallel v_\parallel - n\Omega},$$

where

$$H(v_\parallel) = -\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right) h(v_\parallel) + \frac{k_\parallel^2 v_\parallel^2}{2\omega} h'(v_\parallel)$$
in which \( \omega^2 = 2x T_\perp /m \) and \( h(v) \) is the velocity distribution for motion parallel to \( B_0 \), Eq. (52). Some simple manipulations will show that \( A_n \) and \( B_n \) are related:

\[
B_n = \frac{1}{\omega k_\parallel} \left( \omega - k_\parallel \langle v_\parallel \rangle \right) + \frac{\omega - n\Omega}{k_\parallel} A_n. \tag{60}
\]

For many applications it suffices to evaluate \( \chi \) to lowest or perhaps first order in \( \lambda \sim \rho_L^2 \). The following matrices provide the leading terms:

\[
e^{-\lambda Y_0(\lambda)} \approx \begin{pmatrix}
0 & 0 & \frac{i k_\perp}{2 \Omega} [2 - 3 \lambda] B_0 \\
0 & 2 \lambda A_0 & \frac{2 \omega}{k_\parallel w_1^2} (1 - \lambda) B_0 \\
-\frac{i k_\perp}{2 \Omega} [2 - 3 \lambda] B_0 & \frac{2 \omega}{k_\parallel w_1^2} (1 - \lambda) B_0 & 0
\end{pmatrix}. \tag{61}
\]

\[
e^{-\lambda Y_{\pm 1}(\lambda)} \approx \begin{pmatrix}
(1 - \lambda) A_{\pm 1} & \pm i(1 - 2\lambda) A_{\pm 1} & \pm \frac{k_\perp}{\Omega} (1 - \lambda) B_{\pm 1} \\
\mp i(1 - 2\lambda) A_{\pm 1} & (1 - 3\lambda) A_{\pm 1} & -\frac{i k_\perp}{\Omega} (1 - 2\lambda) B_{\pm 1} \\
\mp \frac{k_\perp}{\Omega} (1 - \lambda) B_{\pm 1} & \frac{ik_\perp}{\Omega} (1 - 2\lambda) B_{\pm 1} & 2(\omega \mp \Omega) \frac{1}{k_\parallel w_1^2} \lambda B_{\pm 1}
\end{pmatrix},
\]

It may be recalled that, just prior to Eq. (42), it was assumed that \( k_y = 0 \). On the other hand, \( k_\perp \) in Eqs. (42)–(63) arises solely from the argument of the Bessel functions in Eq. (43), \( z = k_\perp v_1 /\Omega = (k_x^2 + k_y^2)^{1/2} v_1 /\Omega \), and the susceptibility matrices in (45), (48), and (50) and Eqs. (57)–(63) may be adapted to a differently oriented \( k \) by a simple similarity transformation for the rotation of the \( x \) and \( y \) coordinates:

\[
\chi' = R^{-1} \cdot \chi \cdot R. \tag{64}
\]
In the event that the distribution of parallel velocities is also Maxwellian, the integrals over $H(v_\parallel)$ in Eq. (59) may be evaluated in terms of the plasma dispersion function, $Z_p(\xi_n)$, Eqs. (8-73), (8-74), (8-82), and (8-89)–(8-92). Choosing a shifted Maxwellian as in Eq. (8-69) for the parallel velocity distribution, $h_i(v_\parallel)$,

$$h_i(v_\parallel) = \left[ \frac{1}{\sqrt{\pi} w_\parallel} \exp \left[ - \frac{(v_\parallel - V)^2}{w_\parallel^2} \right] \right]_i,$$

where $V_i = \langle v_i \rangle$, and $w_\parallel^2 = 2\kappa T_i^{(s)}/m_i$, one may find, Prob. 6,

$$A_n = \frac{1}{\omega} \frac{T_1 - T_\parallel}{T_\parallel} + \frac{1}{k_\parallel w_\parallel} \frac{(\omega - k_\parallel V - n\Omega) T_1 + n\Omega T_\parallel}{\omega T_\parallel} Z_0,$$

$$B_n = \frac{1}{k_\parallel} \frac{(\omega - n\Omega) T_1 - (k_\parallel V - n\Omega) T_\parallel}{\omega T_\parallel}$$

$$+ \frac{1}{k_\parallel} \frac{\omega - n\Omega}{k_\parallel w_\parallel} \frac{(\omega - k_\parallel V - n\Omega) T_1 + n\Omega T_\parallel}{\omega T_\parallel} Z_0,$$

$$Z_0 = Z_0(\xi_n), \quad \xi_n = \frac{\omega - k_\parallel V - n\Omega}{k_\parallel w_\parallel},$$

$$\frac{dZ_0(\xi_n)}{d\xi_n} = -2[1 + \xi_n Z_0(\xi_n)].$$

**10-7 Susceptibilities for a Maxwellian $f_0(v_\parallel)$**

The relation for $Z_0'$ is taken from Eq. (8-86). In the event that $V = 0$ and $T_1 = T_\parallel$, $A_n$ and $B_n$ are considerably simplified:

$$A_n = \frac{1}{k_\parallel w_\parallel} Z_0(\xi_n),$$

$$B_n = \frac{1}{k_\parallel} \left[ 1 + \xi_n Z_0(\xi_n) \right] = -\frac{1}{2k_\parallel} \frac{dZ_0(\xi_n)}{d\xi_n}.$$
For convenience, we repeat the power series for $Z_0(\xi)$ from Eqs. (8-76) and (8-88),

$$Z_0(\xi) = i \sqrt{\pi} \text{sgn}(k_\parallel) e^{-\xi^2} - 2\xi + \frac{2 \cdot 2 \xi^3}{3 \cdot 1} - \frac{2 \cdot 2 \cdot 2 \xi^5}{5 \cdot 3 \cdot 1} + \frac{2 \cdot 2 \cdot 2 \cdot 2 \xi^7}{7 \cdot 5 \cdot 3 \cdot 1} - \cdots$$

(68)

and the asymptotic expansion from Eqs. (8-95) and (8-97),

$$Z_0(\xi) \approx i \sigma \sqrt{\pi} \text{sgn}(k_\parallel) e^{-\xi^2} - \frac{1}{\xi} - \frac{1}{2 \xi^3} - \frac{1 \cdot 3}{2 \cdot 2 \xi^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2 \xi^7} - \cdots,$$

(69)

where

$$\sigma = 0 \text{ for } \text{sgn}(k_\parallel) \text{Im}(\xi) = \text{sgn(Im } \omega) > 0,$$

(70)

$$\sigma = 2 \text{ for } \text{sgn}(k_\parallel) \text{Im}(\xi) = \text{sgn(Im } \omega) < 0,$$

but

$$\sigma = 1 \text{ if } |\text{Re } \xi| > 1 \text{ and } |\text{Re } \xi| \cdot |\text{Im } \xi| < \pi/4.$$

For complex values of $k_\parallel$, see the discussion that follows Eq. (8-85).

Depending on the problem at hand, terms from the evaluation of $Z_0$ in Eq. (68) or (69) for each plasma species are to be substituted into the expressions for $A_n$ and $B_n$, Eqs. (66) or (67), and the results in turn used in the $s$-species susceptibilities $\chi_s$, Eq. (57) or Eqs. (61)–(63). Next, the susceptibilities may be combined to form the dielectric tensor:

$$\epsilon = 1 + \sum_s \chi_s$$

(71)
The wave equation for a homogeneous plasma is given by Eq. (1-27),
\[
n \times (n \times E) + \epsilon \cdot E = 0, \quad (72)
\]
or, in matrix form, with \( n_x = 0 \),
\[
\begin{pmatrix}
x_{xx} - n_x^2 & x_{xy} & x_{xz} + n_x n_z \\
x_{yx} & y_{yy} - n_x^2 - n_z^2 & y_{yz} \\
x_{zx} + n_x n_z & y_{zy} & z_{zz} - n_x^2
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix} = 0. \quad (73)
\]
The condition that there be nontrivial solutions from this vector equation is that the determinant of the 3×3 matrix be zero. That condition provides the dispersion relation for the homogeneous-plasma system.

11-2 Propagation Parallel to \( B_0 \)

We restrict our considerations in this chapter to wave propagation in uniform media in which the velocity distributions are Maxwellian in \( (v_{\parallel} - V) \) and in \( v_\perp \), albeit possibly with \( T_\parallel \neq T_\perp \) and with different parameters for the ion and electron components. This prescription still allows a great deal of flexibility in modeling the plasma; in principle, even a loss-cone distribution can be described in this framework by subtracting a low-\( T_\parallel \) Maxwellian from a high-\( T_\parallel \) distribution. [In Chap. 15, however, we will use another quite tractable model: \( f_0(v_{\parallel}) \sim (a + bv_{\parallel}^2) \exp(-v_{\parallel}^2/w_{\parallel}^2) \). Also, see Prob. 10-9.]

The simplest subset of solutions for the hot uniform plasma dispersion relation is that for propagation exactly parallel to the magnetic field, i.e., \( k || B_0 \). In this case the hot-plasma dispersion relation can be factored exactly, just as was the case for parallel propagation in a cold plasma, leading to \( n_\parallel^2 = R, n_\perp^2 = L \) and \( P = 0 \), Eq. (1-37). In the hot-plasma case, Landau and cyclotron damping show up, but because \( k_\parallel = 0 \), finite Larmor radius effects, including cyclotron harmonic damping, do not appear.

From Eqs. (10-61)–(10-63), taking the \( k_\parallel \to 0 \) and \( \lambda \sim k_\parallel^2 \to 0 \) limit, one has just
\[
e^{-\lambda} Y_0(\lambda) = \hat{\epsilon}_\parallel \hat{\epsilon}_\parallel \frac{2\omega}{k_\parallel w_\parallel^2} B_0,
\]
\[ e^{-\lambda Y_{\pm 1}(\lambda)} = \frac{1}{2} \begin{pmatrix} 1 & \pm i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_{\pm 1}, \]

and all other \( Y_n \) are zero. One then finds from Eqs. (10-57), (10-66), (10-71), and (10-73)

\[ n^2 = 1 + \sum_s \frac{\gamma^2}{\omega^2} \left[ \frac{T_1 - T_\parallel}{-T_\parallel} \right] \]

\[ + \frac{(\omega - k_{||} V + \Omega) T_1 - \Omega T_{||}}{k_{||} w_{||} T_{||}} \quad Z_0 \left( \frac{\omega - k_{||} V + \Omega}{k_{||} w_{||}} \right) \left[ \right]_s, \]

\[ n^2 = 1 + \sum_s \frac{\gamma^2}{\omega^2} \left[ \frac{T_1 - T_\parallel}{T_\parallel} \right] \]

\[ + \frac{(\omega - k_{||} V - \Omega) T_1 + \Omega T_{||}}{k_{||} w_{||} T_{||}} \quad Z_0 \left( \frac{\omega - k_{||} V - \Omega}{k_{||} w_{||}} \right) \left[ \right]_s, \]

\[ 0 = \epsilon_{zz}(k_{||}, k_1 = 0) \]
\[ 1 + \sum_s \frac{2\omega_{ps}^2}{k_\parallel w_\parallel^2} \left[ 1 + \frac{\omega - k_\parallel V}{k_\parallel w_\parallel} Z_0 \left( \frac{\omega - k_\parallel V}{k_\parallel w_\parallel} \right) \right]_s, \]

which correspond precisely to \( n_\parallel^2 = R, \ n_\parallel^2 = L, \) and \( 0 = P, \) respectively. One should keep in mind that \( \Omega_s = q_s B_0 / m_s c \) is an algebraic quantity. Collisionfree damping appears when the argument of the dispersion function, \( Z_0(\xi_n), \) is comparable or small compared to unity; thus cyclotron damping occurs for right-handed waves for electrons, and for left-handed waves for ions.

Obtaining the cold-plasma limit of \( Z_0(\xi_{\pm 1}) \) by using just the leading terms in its asymptotic expansion, Eq. (10-69), Eqs. (2) and (3) reduce, away from exact resonance, to

\[ n_\parallel^2 = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega - k_\parallel V_s}{\omega - k_\parallel V_s + \Omega_s}, \]

\[ n_\parallel^2 = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega - k_\parallel V_s}{\omega - k_\parallel V_s - \Omega_s}, \]
which are the cold-plasma forms, Eqs. (1-20), (1-21), and (1-37), slightly modified to include Doppler-shifted susceptibilities and which remain valid even for $T_1 \neq T_\parallel$.

Equation (4) will be found identical to the dispersion relation for one-dimensional plasma oscillations in an unmagnetized Maxwellian plasma, Eq. (8-77), and which was discussed in that context.

Exploring the effects of finite temperature on the $n_\parallel = R, L$ waves, the inclusion of additional terms from the asymptotic expansion of $Z_0(\xi_{\pm 1})$ will bring in the influence of finite parallel electron and ion pressure and of electron ($\xi_{-1}$) and ion ($\xi_{1}$) cyclotron damping. For example, Eqs. (3) and (6) describe the Alfvén torsional mode and the ion cyclotron wave at low frequencies and the QL-L mode at high frequencies. Picking up additional terms in $Z_0(\xi_{1})$ to obtain an improvement over Eq. (6) in the description of the ion cyclotron wave, for the case $V = 0$ and $T_1 = T_\parallel$ for both ions and electrons, and for $0 < \text{Re} \omega < |\Omega_e|$,

$$\frac{k_{\parallel}^2 c^2}{\omega^2} \approx 1 - \omega_{pi}^2 \left[ \frac{1}{\Omega(\omega - \Omega)} + \frac{k_{\parallel}^2 \kappa T}{m \omega(\omega - \Omega)^3} \right]$$

$$- \frac{i \sqrt{\pi}}{\omega |k_{\parallel}^2| w} \exp \left[ - \left( \frac{\omega - \Omega}{k_{\parallel} w} \right)^2 \right] \right],$$

(7)
where $\omega_i^2 = 2\kappa T_i/m_i$. Charge neutrality has been assumed, so that $\omega_p^2/\Omega_i = -\omega_i^2/\Omega_e$. If we separate $\omega = \omega_r + i\omega_i$ and assume that $|\omega_i| \ll \omega_r$, we may use Eq. (7) to derive an approximate value for the damping rate $-\omega_i$:

$$\frac{\omega_i}{\omega_r} \approx - \left[ \frac{\omega^2}{2\Omega^3 - \omega \Omega^2} \frac{\omega_p^2}{k_i^2 c^2} \frac{\sqrt{\pi} \Omega}{k_i w} \exp \left[ \frac{\omega - \Omega}{k_i w} \right] \right]_i.$$  

(8)

This rate may be derived from Eq. (31) in B. N. Gershman (1953), and was also obtained by T. H. Stix (1958) and by J. M. Dawson in J. M. Berger et al. (1958). The numerical evaluation of Eq. (8) is simplified by the evaluation of $\omega_p^2/k_i^2 c^2$ in Eq. (2-27) and by using

$$\frac{\Omega_i}{|k_i w_i|} = \frac{\lambda B_0 Z_i}{905 A_0^{1/2} T_i^{1/2}},$$

where $\lambda = 2\pi k_i^{-1}$ is in centimeters, $B_0$ in gauss, $Z_i$ and $A_i$ are the ion charge and atomic numbers, and $T_i$ is in eV.

11.2 Propagation Parallel to $B_0$

Similarly, if one assumes that $\omega$ is real but that $k_i = k_r + i k_i$, with $|k_i| \ll |k_r|$, the separation of Eq. (7) into real and imaginary parts gives the damping length for a propagating ion cyclotron wave [cf. Eq. (9-23)]:

$$\left( k_i^2 |k| \right) = \left[ \frac{\omega}{2\Omega} \frac{\omega_p^2}{k_i^2 c^2} \frac{\sqrt{\pi} \Omega}{|k_i w|} \exp \left[ \frac{\omega - \Omega}{k_i w} \right] \right]_i.$$  

(10)

For $\omega$ very close to $\Omega_i$, specifically for $\xi_i$ of the order of unity or less, the asymptotic expansion of $Z_0(\xi_i)$ is invalid and Eqs. (5)-(10) do not apply. Returning to Eq. (3), again for $V = 0$, $T_i = T ||$ and $0 < \omega, \ll |\Omega_i|$, one may write

$$n_i^2 = 1 - \frac{\omega_p^2}{\omega \Omega_i} + \left[ \frac{\omega_p^2}{\omega k_i w} Z_0(\xi_i) \right]_i.$$  

(11)
The middle term is the electron contribution. Concerning the ion term, one notes from the graph of the real and imaginary parts of $Z_0(\zeta)/2$ in Fig. 8-14 that $|Z_0(\zeta)/2|$, for $\zeta$ real, never exceeds unity. In fact, this quantity is a maximum at $\zeta = 0$, where it only reaches the value $\sqrt{\pi}/2 = 0.886$. Thus finite temperatures put a maximum value on the magnitude of $|k_\parallel|$, whereas the cold-plasma dispersion relation for ion cyclotron waves, Eq. (6), would allow $k_\parallel^2$ to become infinite at resonance, $\omega = \Omega_i$.

This minimum-wavelength limitation is an important characteristic of plasmas above zero temperature. For ion cyclotron waves, if we neglect the first two terms in Eq. (11), which stem from vacuum displacement current and electron contributions, $|k_\parallel|$ reaches its maximum at $\omega = \Omega_i$ and we may write

$$|k_\parallel|^3 < \frac{\sqrt{\pi} \Omega_i \omega^2}{c^2 w_i}. \quad (12)$$

Similar considerations apply to electron cyclotron waves. For $\omega \gg \Omega_i$, $V_e = 0$, and $T_{\perp}^{(e)} = T_{\parallel}^{(e)}$,

$$\frac{k_\parallel^2 \omega^2}{\omega_e^2} = 1 - \frac{\omega^2}{\omega_e^2} + \left[ \frac{\omega^2}{\omega k_\parallel w} Z_0(\zeta - 1) \right]_e. \quad (13)$$

Neglecting the vacuum and ion contributions, $|k_\parallel|$ reaches its maximum at $\omega = -\Omega_i$:

$$|k_\parallel|^3 < \frac{\sqrt{\pi} |\Omega_e| \omega^2}{c^2 w_e}. \quad (14)$$

11. Waves in Magnetized Uniform Media

Comparing the inequalities for ion and electron cyclotron waves, we note that

$$|k_\parallel|_{\text{max}} \sim \left( \frac{n B_0}{T^{1/2}} \right)^{1/3} \frac{1}{m^{1/2}} \quad (15)$$

so that electron cyclotron waves may have much shorter wavelengths than ion cyclotron waves.
11-3 Cyclotron Harmonic Damping

In calculations of susceptibilities for a magnetized plasma starting with Eq. (10-45), the resonant denominators showed zeros not only at the particle cyclotron frequency, \( \omega = k_\parallel v_\parallel = \Omega_i \), but at each of its integral harmonics, \( n\Omega_i, -\infty < n < \infty \). The physical explanation of the resonances remains the same: in the reference frame of its own zero-order motion, the resonant particle "sees" the first-order electric field at zero frequency. That explanation clearly applies to Landau damping \((n = 0)\), where the Doppler shift alone reduces the laboratory frequency \( \omega \) to zero in the moving particle frame, \( \omega' = \omega - k_\parallel v_\parallel = 0 \). Similarly, for \( k_\parallel = 0 \) and \( \omega = \Omega_i \), the orientation of a left-circularly polarized \( \dot{E} \) field vector remains at a constant angle with respect to the velocity vector for a freely gyrating ion. More generally, resonance at \( \omega' = \omega - k_\parallel v_\parallel - \Omega_i = 0 \) occurs when the Doppler-shifted frequency seen by an ion moving with zero-order velocity \( v_\parallel \) along \( B_0 \) is equal to its own gyrofrequency. Finally, resonance at harmonics of the cyclotron frequency appear when the electric field in the laboratory frame varies not only with \( t \) and \( z \), but also with \( x \) and/or \( y \).

To illustrate second-harmonic cyclotron resonance \((n = 2)\), we may consider an ion gyrating at frequency \( \Omega \) in a magnetic field \( \hat{z}B_0 \). Taking a near optimum case, we choose an electric field that varies as \( \dot{E} = \tilde{y}\sin(k_x x - \omega t) \) with \( \omega = 2\Omega \) and \( k_x = \pi/2\rho_i \). Depicted in Fig. 11-1 are the ion positions at \( \Omega t = 0 \) and \( \Omega t = \pi \). \( \dot{E} \), which is parallel to \( \dot{v} = \tilde{y}v_0 \) at \( t = 0 \), is again parallel to \( \dot{v} \) at \( \Omega t = \pi \), \( \dot{v} = -\tilde{y}v_0 \), at the new ion location. Although the temporal variation, \( \omega t = 2\Omega t = 2\pi \), has restored \( \dot{E}(x,t) \) to its \( t = 0 \) value, at the new ion position, the \( \dot{E} \) field spatial variation, \( k_x \Delta x = \pi \), has reversed the direction of \( \dot{E} \) compared to its direction on the opposite side of the ion orbit.

It is of interest to calculate the rate of energy increase for cyclotron-harmonic resonance, for an ion in a spatially periodic \( \dot{E} \) field. Neglecting any motion parallel to \( B_0 \),

\[
\frac{dW_1}{dt} = \frac{d}{dt} \frac{1}{2} m \dot{v}_1^2 = m \dot{v}_1 \cdot \frac{dv_1}{dt} = q \dot{v}_1 \cdot \left[ \dot{E}_1 + \frac{\dot{v} \times \dot{B}}{c} \right] = q \dot{v}_1 \cdot \dot{E}. \tag{16}
\]

Neglecting changes in the orbit due to the \( \dot{E} \) field, we may write
\[ E = \hat{y}E_{0}\sin(kx - \omega t) \]

\[ \Omega t = 0 \]

\[ \Omega t = \pi \]

\[ E = \hat{x}E \cos(k_{1}x - \omega t) \]

\[ x = \rho_{L} \sin(\Omega t + \phi) \].

Then

\[
\frac{dW_{1}}{dt} = qE\Omega \rho_{L} \cos\left[k_{1}\rho_{L} \sin(\Omega t + \phi) - \omega t\right] \cos(\Omega t + \phi)
\]

\[ = qE\Omega \rho_{L} \text{Re} \left[ \sum_{n=-\infty}^{\infty} J_{n}(k_{1}\rho_{L})e^{i(n\Omega t + \phi)} - i\omega t \right] \text{Re}[e^{i(\Omega t + \phi)}] \]

and it is seen that secular increases (or decreases) in \( W_{1} \) will occur at \( \omega = (n \pm 1)\Omega \) for all \( n \) at rates proportional to \( \rho_{L}J_{n}(k_{1}\rho_{L}) \cos[(n \pm 1)\phi] \).
As an example of the extraction of energy from a wave through cyclotron harmonic interaction, we compute the damping of the fast hydromagnetic wave at \( \omega \approx 2\Omega_i \). This problem might be considered typical of many that arise in uniform-medium hot-plasma waves. Solving them by "brute force" calculations, carrying all the possible relevant terms in the susceptibility tensors, can be very tedious. Instead, in many instances including this one, one may use cold-plasma theory through much of the analysis and introduce the hot-plasma corrections as small perturbations. [An important exception: one should always use the correct form for the reactive part of the electron susceptibility \( \chi'_{ee} \). \( -\omega_{pe}/\omega \) or \((k_{\parallel}^2\Lambda_{e}^2)^{-1} \) or the drift-wave forms, Eqs. (3-52) and (3-53)].

We will assume in this calculation that the damping is weak, that is, \(|\omega| \ll |\omega_i|\). The terms in the ion susceptibility that are going to introduce cyclotron interaction at \( \omega = 2\Omega_i \) are those involving \( A_2 \), Eqs. (10-66), and for a damping calculation we need only the imaginary part. At \( \omega \approx 2\Omega_i \) and for \( \nu_i = 0 \),

\[
\text{Im } A_2 \approx \left[ \frac{\sqrt{\pi}}{|k_{\parallel}||w_{||}|} \exp \left[ - \left( \frac{\omega - 2\Omega_i}{k_{\parallel}w_{||}} \right)^2 \right] \right].
\] (19)

We assume the density is sufficiently large that \( \omega_{pi}^2/\Omega_i^2 \gg 1 \), and note that \( \varepsilon_{zz} \approx -\omega_{pe}^2/4\Omega_i^2 \) will be then much larger than any of the other dielectric tensor elements. Moreover, \( \lambda A_{\pm 2} \) appears in Eq. (10-63) in the same manner as \( A_{\pm 1} \) appears in Eq. (10-62). The significant terms in the wave equation, from Eqs. (10-57), (10-62), (10-63), and (10-73) are then just

\[
\frac{1}{2} \begin{pmatrix}
L_h + R - 2n_i^2 & iL_h - iR \\
-iL_h + iR & L_h + R - 2n_i^2
\end{pmatrix} \begin{pmatrix}
E_x \\
E_y
\end{pmatrix} \approx 0,
\] (20)

where, for \(|\omega| \ll |\omega_i|\),

\[
L_h = L + \Delta L = L + \frac{\omega_{pi}^2}{\omega} \lambda \text{Im } A_2
\] (21)

and where \( R \) and \( L \) are the cold-plasma coefficients Eqs. (1-20) and (1-21).
The cold-plasma dispersion relation in the form given in Eq. (2-19) is helpful in obtaining the dispersion relation from Eq. (20), using \( S = (R + L)/2 \):

\[
\Delta(\omega) \equiv n_i^2 - \frac{(R - n_i^2)(L - n_i^2)}{(S - n_i^2)} - \frac{1}{2} \left( \frac{R - n_i^2}{S - n_i^2} \right)^2 \Delta L = 0. \tag{22}
\]

To solve Eq. (22) for \( \omega_i \), one could differentiate the two reactive terms with respect to \( \omega \), and use \( \omega_i \partial \Delta_i / \partial \omega \approx -\Delta_i \). But a much quicker, albeit approximate, solution can be obtained by first substituting from Eq. (6-38) for the reactive terms in (22):

\[
\omega^2 \Delta(\omega) \approx k_i^2 + \omega + \frac{\Omega_i}{\Omega_i} k_i^2 - \frac{4\pi \rho \omega^2}{B} - \frac{\omega^2}{2c^4} \left( \frac{R - n_i^2}{S - n_i^2} \right)^2 \Delta L = 0.
\tag{23}
\]

One then finds

\[
11-4 \text{ Transit-Time Damping}
\]

\[
\omega_i \approx - \left( \frac{R - n_i^2}{S - n_i^2} \right)^2 \left[ \sqrt{\frac{\pi \rho \omega^2 \lambda}{2|k_\parallel| w_{\parallel} c^2}} \right] \exp \left[ - \left( \frac{\omega^2}{k_\parallel w_{\parallel}^2} \right) \right]. \tag{24}
\]

Laboratory experiments pertaining to wave damping will generally measure the attenuation of the wave in space (the wave being driven by a steady-state oscillator) rather than the wave's attenuation in time. For this application, one might solve Eq. (22) or (23) for \( \text{Im}(k_\parallel) \), for real \( \omega \) and real \( k_i \). For weak damping, the resulting \( \text{Im}(k_\parallel) \) will be related to \( \omega_i \) in Eq. (24) by the group velocity, as in Eq. (9-23).
In retrospect, one might well ask whether the analog of Eq. (24) for \( \omega \approx \Omega_i \) (rather than \( \omega \approx 2\Omega_i \)) would not indicate very strong damping of the fast wave at the fundamental cyclotron frequency. The point is that the \( E \) field becomes circularly polarized in the right-hand direction as \( L \) becomes very large, Eqs. (1-58)–(1-60), and no longer interacts with the ions, whose motion is left-handed. This effect shows up in Eq. (24) in the \((S - n_\parallel)^2\) term in the denominator, although in this context \( S \) would no longer be purely reactive but would include both the real and imaginary parts of \( \varepsilon_{xx} \). And in very hot plasmas, it may also be necessary to correct Eq. (24) for this same reason; that is, if \( \Delta L \) due to \( \omega = 2\Omega_i \) resonance starts to be comparable in magnitude to the cold-plasma \( R \) and \( L \).

On the other hand, absorption at the fundamental cyclotron frequency can be important if the resonant ions comprise only a small minority of the total ion component. Then their contribution to \( S \) is not overwhelming. Radiofrequency heating schemes often take advantage of this fact.

### 11-5 Propagation Perpendicular to \( B_0 \), \( \omega \neq n\Omega \)

In treating waves in a cold plasma, we were able to find immediate factorings of the dispersion relation for propagation both parallel and perpendicular to the magnetic field. For parallel propagation, the cold-plasma relations \( n_\parallel^2 = R \), \( n_\parallel^2 = L \) and \( P = 0 \), Eq. (1-37), found precise analogs in hot-plasma theory, in Eqs. (2)–(4). For exact perpendicular propagation, \( n_\parallel^2 = RL/S \) and \( n_\parallel^2 = P \), Eq. (1-38), one may also find hot-plasma analogs but a possible ambiguity appears that complicates the situation. The complication arises in taking the simultaneous limits for \( k_\parallel = 0 \) (perpendicular propagation) and \( \omega = n\Omega \) for integer \( n \). Postponing the problem, we look first at the case for \( V = 0, k_\parallel \to 0, \omega \neq n\Omega \). In this limit, the argument of the plasma dispersion function, \( Z_0(\xi) \), diverges, \( \xi_n = (\omega - k_\parallel V - n\Omega)/k_\parallel \omega \to \pm \infty \). The coefficients \( A_n \) and \( B_n \) in the dielectric tensor [Eqs. (10-57) and (10-66)] then become, to first order in \( k_\parallel \), with \( V = 0 \) and \( \omega \neq n\Omega \),

\[
A_n \to -\frac{1}{\omega - n\Omega},
\]

\[
B_n \to -\frac{k_\parallel}{2(\omega - n\Omega)^2} \left[ \frac{n\Omega}{\omega} \right],
\]

\[
B_n \to -\frac{k_\parallel}{2(\omega - n\Omega)^2} \left[ \frac{n\Omega}{\omega} \right].
\]
Substituting from Eq. (32) into Eq. (10-57), one finds, again for $V = 0$ and $\omega \neq n\Omega$,

$$
\chi_{xx} = -\frac{\omega^2}{\omega} \sum_{n = -\infty}^{\infty} e^{-\lambda} \frac{n^2 I_n(\lambda)}{\lambda} \frac{1}{\omega - n\Omega},
$$

$$
\chi_{xy} = -\chi_{yx} = i \frac{\omega^2}{\omega} \sum_{n = -\infty}^{\infty} n e^{-\lambda} \left[ I_n(\lambda) - I_n'(\lambda) \right] \frac{1}{\omega - n\Omega},
$$

$$
\chi_{yy} = -\frac{\omega^2}{\omega} \sum_{n = -\infty}^{\infty} e^{-\lambda} \left[ \frac{n^2}{\lambda} I_n(\lambda) + 2\lambda I_n'(\lambda) \right]
$$

(33)

\begin{align*}
11-5 & \text{ Propagation Perpendicular to } B_0, \omega \neq n\Omega \\
& - 2\lambda I_n'(\lambda) \frac{1}{\omega - n\Omega},
\end{align*}

$$
\chi_{zz} = -\frac{\omega^2}{\omega} \sum_{n = -\infty}^{\infty} e^{-\lambda} I_n(\lambda) \left( 1 - \frac{n\Omega}{\omega} \frac{T_1 - T_\parallel}{T_1} \right) \frac{1}{\omega - n\Omega},
$$

and, for $k_1 \to 0$, all other $\chi_{ij} = 0$. In fact, provided that $f_0(p_1, p_\parallel)$ is even in $p_1$, it is trivial to show, even relativistically, that $\chi_{xx} = \chi_{zz} = 0$ and $\chi_{xy} = -\chi_{yx} = 0$ when $k_1 = 0$. The point is simply that for $k_1 = 0$, since $\Omega = \Omega(p_1^2 + p_\parallel^2)$ and $U \to \partial f_0 / \partial p_1$, the integrand in Eq. (10-48) is odd in $p_\parallel$ and the integral is zero.
Since \( n_{\parallel} = 0 \) in this limit, the wave equation in Eq. (10-73) with \( \chi \) from Eq. (33) factors into two equations, one for the \( x,y \) manifold and one for the \( z \) direction. Their respective dispersion relations are

\[
\begin{align*}
\eta^2_x &= \epsilon_{yy} - \frac{\epsilon_{yx}\epsilon_{yx}}{\epsilon_{xx}}, \\
\eta^2_x &= \epsilon_{zz}.
\end{align*}
\]  

(34)

(35)

corresponding to \( n_1^2 = S - D^2/S = (S^2 - D^2)/S = RL/S \) and \( n_1^2 = P \) in the cold-plasma case, Eqs. (1-33) and (1-38). And because \( \epsilon_{xx}, \epsilon_{yz}, \epsilon_{yx} \), and \( \epsilon_{yz} \) vanish at \( k_{\parallel} = 0 \) even in the \( f_0(p_x, p_y) \) relativistic case, the factoring into Eqs. (34) and (35) remains valid also for this situation. On the other hand, we will see in the next section that this convenient factoring breaks down for very small but finite \( k_{\parallel} \), namely \( k_{\parallel} \omega_{\parallel} \sim \omega - n \Omega \).

While the cold-plasma dispersion relations are virtually unaffected by wave-particle interaction at the cyclotron fundamental [\( S = (R + L)/2, n_1^2 \rightarrow 2L \) and \( 2R \) at the right-handed and left-handed resonances, respectively], the susceptibilities in Eq. (33) show that hot-plasma waves can be strongly influenced not only by the fundamental but also by harmonic gyrofrequencies. And since for \( k_{\parallel} = 0 \) there is no damping away from exact resonance, \( \omega = n \Omega \), the reactive near-resonance effects are especially evident in Eqs. (34) and (35).

Equation (34) describes an electromagnetic wave for which the \( E \) fields lie entirely in the \( x,y \) plane. The relative polarization of \( E_x \) and \( E_y \) may be found from the top line of the wave equation (10-73):

\[
\frac{iE_x}{E_y} = -i\frac{\epsilon_{xy}}{\epsilon_{xx}}.
\]

\(  \)  

(36)
If $|\varepsilon_{xx}| < |\varepsilon_{yy}|$, E will lie in the same direction as $\mathbf{k} = \hat{z}k_z$, implying that it can be represented as $E \approx -\nabla \phi$, that is, the wave is approximately electrostatic. This subclass of Eq. (34), called Bernstein modes, will be analyzed in some detail in Sec. 10.

For the modes in Eq. (35), the E field is always parallel to $B_0$ while k is perpendicular to $B_0$. Thus these modes are electromagnetic, even near cyclotron or cyclotron harmonic resonance, and cannot become electrostatic. Another feature that distinguishes them from the Bernstein modes is that a gap in the spectrum for propagation near resonance for the electromagnetic cyclotron harmonic modes in Eq. (35) always occurs for $|\omega|$ slightly larger than $|n\Omega|$, whereas the converse will be seen to hold for the Bernstein modes, Eq. (87) below.

Finally, we might note that for $\omega \sim n\Omega$, the electron contribution to $\varepsilon_{zz}$ in Eq. (35) is $\chi^{(e)}_{zz} = -\frac{\omega_p^2}{\omega^2}$. The mode will be evanescent, then, everywhere but so close to the ion resonances that $\chi^{(i)}_{zz}$ exceeds $\frac{\omega_p^2}{\omega^2}$. However, a finite $k_\parallel$ will limit the magnitude of $\chi^{(e)}_{zz}$ to $\sim \frac{\omega_p^2}{\omega^2} k_\parallel |\omega^{(i)}_{0}|$.

### 11.6 Propagation Approximately Perpendicular to $B_0$, $\omega \sim n\Omega$

The argument of the plasma dispersion function, $Z_0(\xi_n)$, is $\xi_n = (\omega - k_\parallel V - n\Omega)/k_\parallel \omega$ and if, for $V = 0$, one lets both $k_\parallel \to 0$ and $\omega - n\Omega \to 0$, then $\xi_n \to 0/0$ and is indeterminate. One way around the difficulty is to introduce the relativistic variation of mass, hence of $\Omega$, with energy. Then in forming moments of $f_1(k, \omega, p)$, the resonant denominator still varies with both $p_\parallel$ and $p_\perp$, Eqs. (10-35), (10-44), and (10-45), and the relativistic dispersion function is well defined, even for $k_\parallel = 0$. This approach to the problem is examined in the latter half of this section. Alternatively, one may resolve the question on the basis of causality or, equivalently, by introducing very weak collisions modeled by $\omega \sim \omega + iv$, $v > 0$, Eqs. (8-33)–(8-35). It simplifies the considerations if we again set $V = 0$ and also $T_\perp = T_\parallel$. Then $A_n$ and $B_n$, from Eq. (10-67), are

\[
A_n = \frac{1}{k_\parallel \omega} Z_0(\xi_n) - \frac{1}{\omega + iv - n\Omega} \\
= - \left[ p \left( \frac{1}{\omega - n\Omega} \right) - i\pi \delta(\omega - n\Omega) \right],
\]

\[
B_n = - \frac{1}{2k_\parallel} Z_0(\xi_n) - \frac{1}{2} \frac{k_\parallel \omega^2}{(\omega + iv - n\Omega)^2} \\
= \frac{k_\parallel \omega^2}{2} \frac{\partial}{\partial \omega} \left[ p \left( \frac{1}{\omega - n\Omega} \right) - i\pi \delta(\omega - n\Omega) \right],
\]
\[
\frac{2\xi_n}{w} B_n = -\frac{1}{k_\parallel w} \xi_n Z'(\xi_n) \to -\frac{1}{\omega + iv - n\Omega} \approx - \left[ P\left(\frac{1}{\omega - n\Omega}\right) - i\pi \delta(\omega - n\Omega) \right].
\]

These equations are in accord with the corresponding relations for \( A_n \) and \( B_n \) in the previous section, Eq. (32), but also with the Plemelj prescription for causality, Eq. (3-20). The ambiguity in approaching the simultaneous limits of \( k_\parallel \to 0 \) and \( \omega - n\Omega \to 0 \) is now resolved provided one keeps \( v > 0 \) finite during the limiting process for \( k_\parallel \) and \( \omega \). In passing, we may note that \( \xi_n = (\omega + iv - n\Omega)/k_\parallel w \to iv/k_\parallel w \to i\infty \text{sgn}(k_\parallel) \), in which case \( \sigma = 0 \) in the asymptotic expansion of \( Z_\infty(\xi_n) \), Eq. (8-97) or (10-69).
Using Eq. (37) in the susceptibility tensor Eq. (10-57) leads again to the set (33) for the precise case \( k_\parallel = 0 \), provided \( \omega - n\Omega \) is replaced by \( \omega + \nu - n\Omega \). But nearby one also finds

\[
\chi_{xz} = \chi_{zx} = - \frac{\omega_p^2}{2\omega k_\parallel\Omega} \sum_{n = -\infty}^{\infty} e^{-\lambda} \frac{n I_n(\lambda)}{\lambda} Z'_0(\xi_n),
\]

\[
\chi_{yz} = - \chi_{zy} = - \frac{i\omega_p^2}{2\omega k_\parallel\Omega} \sum_{n = -\infty}^{\infty} e^{-\lambda}[I_n(\lambda) - I'_n(\lambda)] Z'_0(\xi_n),
\]

(38)

\[
Z'_0(\xi_n) \approx \frac{k_\parallel^2 \omega^2}{(\omega + i\nu - n\Omega)^2}.
\]

Thus for small but finite \( k_\parallel \), as mentioned in the previous section, these terms—with their resonant denominators—can become very large. The convenient \( k_\parallel = 0 \) factoring of the dispersion relation into Eqs. (34) and (35) is then no longer generally valid.

On the other hand, near cyclotron resonance, for small \( k_\parallel \) and for small \( k_\parallel \rho_L \) it is still possible to obtain an approximate factoring of the dispersion relation. We start from the determinant of the matrix in (10-73), drop the terms that are explicitly \( n_x^2 \) or \( n_x n_z \), and make use of the Onsager relations \( \epsilon_{yx} = -\epsilon_{xy}, \epsilon_{zx} = \epsilon_{xz}, \) and \( \epsilon_{yz} = -\epsilon_{zy} \) [cf. Eqs. (10-45) and (10-48)]. Then, after multiplying each term of the determinant by \( \epsilon_{xx} \), there results

\[
(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx} - \epsilon_{xx}n_x^2)(\epsilon_{zz}\epsilon_{xx} - \epsilon_{zx}\epsilon_{xz} - \epsilon_{xx}n_x^2)
\]

\[
= -(\epsilon_{xx}\epsilon_{yx} + \epsilon_{xy}\epsilon_{xz})^2 \approx \epsilon_{xz}\epsilon_{xz}(\epsilon_{xx} \pm i\epsilon_{xy})^2.
\]

The approximation \( \epsilon_{yz} = \mp i\epsilon_{xz} \) comes from looking at just the resonant terms in Eq. (38) for which, if \( \lambda \ll 1 \), it holds that \( \epsilon_m = -i\epsilon_{yz} = i\epsilon_{xy} \).
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\[ \approx \mp \varepsilon_{xz} = \mp \varepsilon_{zx} \]. The upper and lower signs refer to ion resonance \((n > 0)\) and electron resonance \((n < 0)\), respectively. A portion of the factor of \(\varepsilon_{xz}\varepsilon_{zx}\) on the left is \(\varepsilon_{xx} \varepsilon_{yy} - \varepsilon_{xy} \varepsilon_{yx} = \varepsilon_{xx}^2 + \varepsilon_{yy}^2 = (\varepsilon_{xx} + i \varepsilon_{xy}) (\varepsilon_{xx} - i \varepsilon_{xy}) \). But on the right, the factor for \(\varepsilon_{xz}\varepsilon_{zx}\) is \((\varepsilon_{xx} \pm i \varepsilon_{xy})^2\) and the choice is always such (+ for \(n > 0\), ions, and - for \(n < 0\), electrons) that the resonant portions almost cancel. This term may therefore be neglected, leading to a factored dispersion relation. The first factor is exactly Eq. (34) corresponding, for cold plasmas, to the extraordinary mode \(n_1^2 = RL/S\). The second factor can be expressed in a form akin to Eqs. (34) and (60) below, namely,

\[ n_x^2 \approx \varepsilon_{zz} - \frac{\varepsilon_{xx} \varepsilon_{xz}}{\varepsilon_{xx}}, \tag{39} \]

which reduces to Eq. (35) when \(k_\parallel = 0\) precisely.

Exploring Eq. (39) near electron cyclotron resonance, \(\omega = - \Omega_e\), we find from Eqs. (38), (10-57), (10-62), and (10-67) for \(\lambda\) small, \(V = 0\), \(T_\perp = T_\parallel\),

\[ \varepsilon_{xx} = \left[ \frac{\omega_p^2}{2 \omega k_\parallel w} Z_0'(\xi - 1) \right]_e, \]

\[ \varepsilon_{xz} = \varepsilon_{zx} \approx - i \varepsilon_{yz} = i \varepsilon_{zy} = \left[ \frac{\omega_p^2 k_\perp}{4 \omega \Omega k_\parallel} Z_0'(\xi - 1) \right]_e, \]

\[ \varepsilon_{zz} = P - \left[ \frac{\omega_p^2 k_\perp^2 w}{4 \omega \Omega k_\parallel^2} \xi - 1 Z_0'(\xi - 1) \right]_e, \tag{40} \]

\[ P = 1 - \frac{\omega_p^2}{\omega^2}, \quad \xi - 1 = \frac{\omega + \Omega}{k_\parallel w}. \]
Substituting into Eq. (39) from Eq. (40) and using Eq. (10-67) twice to evaluate $Z'_0$, one obtains simply

$$n_x^2 \left[ 1 + \frac{\omega_p^2 \omega}{2 \Omega^2 n || c} \left[ \xi - 1 + \frac{1}{Z_0(\xi - 1)} \right] \right]_e$$

$$= n_x^2 \left[ 1 - \frac{\omega_p^2 \omega}{4 \Omega^2 n || c} \frac{Z_0'(\xi - 1)}{Z_0(\xi - 1)} \right]_e = P.$$  \hspace{1cm} (41)

Now, using just the leading asymptotic terms in $Z_0$ but retaining a finite-collision frequency, $k_||$ actually disappears and Eq. (41) becomes just

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$$n_x^2 \left[ 1 + \frac{\beta_e}{4} \frac{\omega}{\omega + i \nu + \Omega_e} \right] = P,$$ \hspace{1cm} (42)

where $\beta_e = 8 \pi n_e \kappa T_e / B_0^2$. Equation (42) also follows from Eq. (39) even if the second term on the right is omitted, as in Eq. (35). On the other hand, as Eq. (42) results from the asymptotic expansion of $Z_0(\xi - 1)$, it should not be expected to hold where $|\omega + \Omega_e| \approx |k_|| w_e|$ unless the actual collision rate is such that $\nu \gtrsim |k_|| w_e|$. But in the latter circumstance [and subject to the inequality for $\nu$ in Eq. (45), below] one may use Eq. (42) to evaluate the integrated absorption for a wave passing through a slab of plasma somewhere in the interior of which $\omega = - \Omega_e(r)$. Then for $\beta_e \ll 1$, writing $n_x = n_{xr} + i n_x^i$ and balancing the imaginary parts in Eq. (42),
\[ \text{Im} \int_{x_1}^{x_2} dx \ k_x(k_{||}, \omega, \Omega) = \text{Im} \frac{1}{d\Omega/dx} \int_{\Omega_1}^{\Omega_2} d\Omega \ k_x(k_{||}, \omega, \Omega) \]
\[ = \frac{k_x \omega \omega_{pe}^2 \omega_e^2}{8 \Omega_e c^2} \frac{\nu}{d\Omega_e / dx} \int d\Omega_e (\omega + \Omega_e)^2 + \nu^2 \]
\[ = \left| \frac{\pi k_x \omega_{pe}^2 \omega_e^2}{8 c^2 \Omega_e d\Omega_e / dx} \right| \]

(43)

for \( x_2 > x_{\text{resonant}} > x_1 \).

Moreover, as long as \( k_{||} \) remains finite [actually, provided \(|n_{||}| \gg \omega_e/c\), as explained below, Eq. (45)], the integrated absorption can be evaluated from the more exact dispersion relation (41) even for the collisionfree case. Again for \( \beta_e \ll 1 \),

\[ \text{Im} \int_{x_1}^{x_2} dx \ k_x(k_{||}, \omega, \Omega) = \text{Im} \frac{k_{||} \omega \omega_{pe}^2 \omega_e^2}{c \ d\Omega_e / dx} \int d\xi - 1 n_x \]
\[ \approx \text{Im} \frac{k_{||} \omega \omega_{pe}^2 \omega_e^2}{c \ d\Omega_e / dx} \frac{n_x \omega_{pe}^2 \omega_e}{8 \Omega_e c n_{||}} \]
\[ \times \int d\xi - 1 \frac{Z_0(\xi - 1)}{Z_0(\xi - 1)} \]

\[ = \left| \frac{\pi k_x \omega_{pe}^2 \omega_e^2}{8 c^2 \Omega_e d\Omega_e / dx} \right| , \]  

(44)
exactly as in Eq. (43). The integral in Eq. (44) is just \( \int dZ_0 / Z_0 = -i\pi \) with \( \xi_{-1} \) running from \(-\infty\) to \(\infty\). The result for integrated absorption seen in Eqs. (43) and (44) was obtained by T. M. Antonsen, Jr. and W. M. Manheimer (1978).

Although neither \( k_\parallel \) nor \( n_\parallel \) appear explicitly in any of the last three equations, there is still a problem with the ambiguity in the determination of \( \xi_{-1} \) when \( \omega \rightarrow -\Omega_e \) and \( k_\parallel \rightarrow 0 \) simultaneously, in which event \( \xi_{-1} \rightarrow 0/0 \). As suggested in the first paragraph of this section, the question may be resolved by taking into account the relativistic increase of mass, leading to \( \Omega \rightarrow \Omega_0 (1 - v^2/c^2)^{1/2} \), Eq. (10-35). Equation (42) may be used to elucidate the parameter range subject to this correction. We substitute \( \nu_{\text{eff}} \sim k_\parallel w_e \) in a crude representation for the natural width of the collisionfree resonant interaction. Then \( \nu_{\text{eff}} \) will always be large compared to the relativistic correction provided

\[
\nu_{\text{eff}} \sim k_\parallel w_e \gg \Omega_e w_e^2 / 2c^2 \quad \text{or} \quad |n_\parallel| \gg w_e / c. \tag{45}
\]
Inequality (45) therefore offers validity criteria for Eqs. (41)–(44), either in terms of an actual collision frequency $\nu$ or a finite wave number $k_\parallel$.

Another point of interest concerning Eq. (42) is the resonance, $n_x^2 \to \infty$, occurring at $(\omega + \Omega_e)/\omega \approx -\beta_e/4$, that is, for $|\omega|$ just slightly smaller than $|\Omega_e|$. Now $\beta_e = 8\pi n_e\kappa T_e / B_0^2 = (\omega_{pe}^2/\Omega_e^2)(w_e/c^2)$, and $\omega_{pe}^2/\Omega_e^2$ is of order unity or less for many laboratory and astrophysical plasmas. Thus the resonance would occur where the fractional frequency deviation away from $|\Omega_e|$ would be smaller than $w_e^2/c^2$, but the relativistic corrections to $\Omega_e$ are just of this order and Eqs. (41) and (42) must be corrected to describe this circumstance with accuracy.

The most important relativistic corrections, for $\nu^2/c^2 \ll 1$, are not difficult to evaluate for the case of small $k_\perp \rho_L$ and exact perpendicular propagation, $k_\parallel = 0$. We focus again on $\omega \simeq -\Omega_e$ for a plasma with $V_e = 0$ and $T_{\perp}^{(e)} = T_{\parallel}^{(e)}$ and modify Eqs. (10-44) and (10-48) to incorporate the critical correction to $\Omega$ for small $\nu^2/c^2$,

$$f_0(v) = \frac{1}{\pi^{3/2}w^3} \exp\left(-\frac{v_\perp^2 + v_\parallel^2}{w^2}\right),$$

$$p \rightarrow m_0v,$$  \hspace{1cm} (46)

$$\omega - n\Omega_e(v) = \omega - n\Omega_{e0}\left(1 - \frac{\nu^2}{c^2}\right)^{1/2} \approx \omega - n\Omega_{e0}\left(1 - \frac{v_\perp^2 + v_\parallel^2}{2c^2}\right).$$

For $k_\perp \rho_L \ll 1$ and $\Re \omega > 0$, using Eq. (10-38) and noting that $f_0(v)$ in Eq. (46) is isotropic, the $n = -1$ contribution to the susceptibility tensor in Eq. (10-48) becomes
\[ \chi_{e}^{(-1)} = -i \frac{\omega_{pe}^2}{\omega} \int_{0}^{\infty} d\tau \int_{0}^{\infty} 2\pi v_{\perp} dv_{\perp} \int_{-\infty}^{\infty} dv_{\parallel} e^{i(\omega - k_{\parallel} v_{\parallel} + \Omega_{e}(v)) \tau} \times T_{-1} v_{\perp} \frac{\partial f_{0}(v)}{\partial v_{\perp}}, \] (47)

where \( \alpha \equiv k_{\perp} v_{\parallel} / \Omega_{e} \). Substituting for \( \Omega_{e}(v) \) and \( f_{0}(v) \) from Eq. (47), the \( v_{\perp} \) integral in Eq. (47) is

\[ I = -2 \int_{0}^{\infty} dx x e^{-x(1 - i\tau/\tau_{0})} \]
\[ K_n = \frac{1}{\pi^{1/2} \omega} \int_{-\infty}^{\infty} dv_{\parallel} e^{-i(k_{\parallel}v_{\parallel} + \Omega_e v_{\parallel}^2/2\omega)e} e^{-\frac{v_{\parallel}^2}{\omega^2}} \left( \frac{k_{\parallel}v_{\parallel}}{\Omega_e} \right)^n, \]

where \( x = v_{\parallel}^2/\omega^2 \) and \( \tau_0 = |2c^2/\omega^2\Omega_e| \). Next, the integral over \( v_{\parallel} \) (recall that \( \alpha = k_{\parallel}v_{\parallel}/\Omega_e \)),

\[ K_0 = \frac{1}{(1 - i\tau/\tau_0)^{1/2}} \text{ for } k_{\parallel} = 0, n = 0, \]

\[ K_2 = \frac{1}{2\left( \frac{k_{\parallel}\omega}{\Omega_e} \right)^2} \frac{1}{(1 - i\tau/\tau_0)^{3/2}} \text{ for } k_{\parallel} = 0, n = 2. \]

The large advantage in restricting the analysis to \( k_{\parallel} = 0 \) becomes evident at this point: although the integral for \( K_n \) is easily performed for \( k_{\parallel} \neq 0 \), by completing the square in the exponent, the result produces a very difficult integral over \( \tau \) in the next step. Staying then with \( k_{\parallel} = 0 \), we find the electron susceptibilities, for \( \text{Re } \omega > 0 \) and \( \lambda_e \ll 1 \),

\[ \chi_{xx}^{(-1)} = \chi_{yy}^{(-1)} = i\chi_{xy}^{(-1)} = \chi_{yx}^{(-1)} = \text{ } \]

\[ \chi_{xx}^{(-1)} = \chi_{yy}^{(-1)} = -i\chi_{xy}^{(-1)} = \chi_{yx}^{(-1)} = \text{ } \]

\[ \chi_{zz}^{(-1)} = \frac{i\omega^2}{4\omega} \left( \frac{k_{\parallel}\omega}{\Omega_e} \right)^2 \int_0^{\infty} d\tau \frac{e^{i(\omega + \Omega_e)\tau}}{(1 - i\tau/\tau_0)^{7/2}}. \]

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\[ = i \frac{\omega^2}{2\omega} \int_0^{\infty} d\tau \frac{e^{i(\omega + \Omega_e)\tau}}{(1 - i\tau/\tau_0)^{5/2}}, \]

\[ \chi_{xx}^{(-1)} = \chi_{yy}^{(-1)} = -i\chi_{xy}^{(-1)} = i\chi_{yx}^{(-1)} = 0, \]

\[ \chi_{zz}^{(-1)} = \frac{i\omega^2}{4\omega} \left( \frac{k_{\parallel}\omega}{\Omega_e} \right)^2 \int_0^{\infty} d\tau \frac{e^{i(\omega + \Omega_e)\tau}}{(1 - i\tau/\tau_0)^{7/2}}. \]
The corresponding susceptibilities for $\omega + 2\Omega_e \simeq 0 \ (n = -2)$ are similar to those in Eq. (50). From Eq. (10-57) or, more simply, just by comparison of Eqs. (10-62) and (63), one sees that the $\chi^{(-2)}_e$ are smaller than the $\chi^{(-1)}$ by the factor $\lambda_e$. The one exception is that $\chi^{(-2)}_{zz}$ is down from $\chi^{(-1)}_{zz}$ by the factor $\lambda_e/4$. Moreover, $\omega + \Omega_e$ in Eq. (50) must be replaced everywhere by $\omega + 2\Omega_e$.

Comprehensive treatments of the dielectric tensor for relativistic Maxwellian plasmas have been given by B. Trubnikov (1959); Yu. N. Dnestrovskii, D. P. Kostomarov, and N. V. Skrydlov (1964); I. P. Shkarofsky (1966); and M. Bornatici, R. Cano, D. DeBarbieri, and F. Engelmann (1983). In the notation of Dnestrovskii et al. and Shkarofsky, Eq. (50) would read

\[
\chi_{xx}^{(-1)} = \chi_{yy}^{(-1)} = i\chi_{xy}^{(-1)} = -i\chi_{yx}^{(-1)} = -\frac{\omega^2_{pe}}{2\omega} \tau_0 F_{5/2} \left( \frac{\mu \delta \omega}{\Omega_0} \right),
\]

\[
\chi_{zz}^{(-1)} = -\frac{\omega^2_{pe}}{4\omega} \left( \frac{k_1 w}{\Omega_e} \right)^2 \tau_0 F_{7/2} \left( \frac{\mu \delta \omega}{\Omega_0} \right), \quad (51)
\]

where

\[
\mu \equiv \frac{m_e c^2}{\kappa T_e} \to \frac{2c^2}{w_e^2}, \quad \delta \equiv \frac{\omega + \Omega_e}{\omega}, \quad \tau_0 = \frac{2c^2}{w^2 \Omega_0}, \quad (52)
\]

and where, with $t = \tau/\tau_0$,

\[
F_q(z) = -i \int_0^\infty dt \frac{e^{izt}}{(1 - it)^q}. \quad (53)
\]
Integration by parts will confirm the recursion formula for $F_q(z)$, valid for $\text{Im } z > 0$,

$$(q - 1)F_q(z) = 1 - z F_{q - 1}(z) \quad (54)$$

and it is not difficult to verify the useful relation found by Shkarofsky, Prob. 1 and Eq. (8-80),

$$F_{1/2}(z) = -\frac{i}{\sqrt{z}} Z(i \sqrt{z}). \quad (55)$$

Analytic continuation can then be used to extend Eq. (54) to the lower half of the $z$ plane and to evaluate $\chi^{(-1)}$ in Eq. (51) in terms of the plasma dispersion function, Eq. (8-80). Using Eqs. (54) and (55) together with the asymptotic evaluation of $Z(\zeta)$ in Eq. (10-69), one finds, Probs. 2 and 3:

$$F_{5/2}(z) = \frac{2}{3} - \frac{4}{3} z - \frac{4}{3} i z^{3/2} Z(i \sqrt{z}) \approx \frac{1}{z} - \frac{5}{2 z^2} + \cdots$$

$$-i \sigma \frac{4}{3} \sqrt{\pi} |z|^{3/2} e^{-|z|}, \quad (56)$$
\[
F\frac{7}{2}(z) = \frac{2}{5} - \frac{4}{15} z + \frac{8}{15} z^2 + \frac{8}{15} i z^{5/2} Z(i \sqrt{z}) \approx \frac{1}{z} - \frac{7}{2z^2} + \cdots \\
- i \sigma \frac{8}{15} \sqrt{\pi} |z|^{5/4} e^{-|z|},
\]

(57)

where, in this application, \( \sigma = 0 \) for \( z \sim \omega + \Omega_e \lambda > 0 \) and \( \sigma = 1 \) for \( z \sim \omega + \Omega_e \lambda < 0 \). Looking back at the susceptibilities in Eq. (51), one sees now that there is zero damping for \( |\omega| > |\Omega_e \lambda| \), but finite damping for \( |\omega| < |\Omega_e \lambda| \). The physical explanation is simply that the relativistic change of mass only lowers the cyclotron frequency, so single particle resonance in the absence of a Doppler shift (since \( k_{\parallel} = 0 \)) can occur only at a frequency below \( |\Omega_e \lambda| \).

Graphs of the imaginary parts of \( F_{5/2}(z) \) and \( F_{7/2}(z) \) are shown in Fig. 11-2. For the ordinary mode, \( n_z^2 = \varepsilon_{zz} \approx P + \chi_{zz}'(-1) \), the damping will peak near \( z = -\frac{5}{2} \), or \( (\omega + \Omega_e \lambda)/\Omega_e \lambda = \frac{5}{2} k T_e / m_0 c^2 \), and the half-maximum points are \( z = -1.069 \) and \( z = -4.85 \).

From Eqs. (51) and (57), it is not difficult to show that the ordinary-mode integrated absorption in the weakly relativistic case leads once again to the result given in Eqs. (43) and (44), Prob. 4, K. R. Chu and B. Hui (1983).