Formal Theory of MHD Stability: Energy Principle

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Outline

• Definition of stability

• Linear stability formulation as
  – initial value problem
  – normal mode eigenvalue problem
  – variational principle
  – Energy Principle

• Classification of ideal MHD instabilities

• Example: stability of $\theta$ pinch

Preliminary Remarks

• Ideal MHD Equations \((d/dt \equiv \partial/\partial t + V \cdot \nabla)\):

\[
\begin{align*}
\frac{d\rho}{dt} + \rho \nabla \cdot V &= 0, \quad \frac{dp}{dt} + \gamma p \nabla \cdot V = 0 \\
\rho \frac{dV}{dt} &= c^{-1} J \times B - \nabla p, \quad E + c^{-1} V \times B = 0 \\
\frac{\partial B}{\partial t} + c \nabla \times E &= 0, \quad \nabla \times B = (4\pi/c) J, \quad \nabla \cdot B = 0
\end{align*}
\]

• Assume there exists a static equilibrium:

\[
V_0 = E_0 = 0, \quad c^{-1} J_0 \times B_0 = \nabla p_0, \quad \nabla \times B_0 = (4\pi/c) J_0, \quad \nabla \cdot B_0 = 0
\]

• Is it stable or unstable? Linear stability analysis is usually sufficient, no need for non-linear studies

• Reason: unstable ideal MHD modes usually destroy the plasma \(\Rightarrow\) have to avoid them, not study the details of plasma self-destruction
Definition of Stability

• **Linearize all quantities** of interest about their equilibrium values

\[ q(r, t) = q_0(r) + \tilde{q}_1(r, t), \quad \tilde{q}_1(r, t) = q_1(r) \exp(-i\omega t), \quad |\tilde{q}_1|/|q_0| \ll 1 \]

• **By definition**, 

\[ \text{Im}(\omega) > 0 \quad \Rightarrow \quad \text{(exponential) instability} \]
\[ \text{Im}(\omega) \leq 0 \quad \Rightarrow \quad \text{(exponential) stability} \]

• **Word of caution**: this definition implicitly assumes that the **modes are discrete**.

• **They are not necessarily** in the stable part of the ideal MHD spectrum (continuum of modes may exist), but **they are** in the unstable part of the spectrum for all systems studied so far.
Energy Principle: Plan of Attack

- **Energy Principle** is a powerful method for testing ideal MHD stability in arbitrary 3D magnetic confinement configurations.

- Will derive it by going through the following steps:
  - Formulate the stability problem as an *initial value problem* for plasma displacement from an equilibrium position.
  - Cast the initial value problem in the form of a *normal-mode eigenvalue problem*.
  - Transform the eigenmode formulation into a *variational form*.
  - Reduce the variational formulation to the **Energy Principle**.
Initial Value Formulation

• Assume a static ideal MHD equilibrium and linearize all quantities about it:

\[ q(r, t) = q_0(r) + \tilde{q}_1(r, t) \]

• Introduce plasma displacement vector \( \tilde{\xi} \) defined as \( \tilde{V}_1 \equiv \partial \tilde{\xi}/\partial t \)

• Choose initial conditions:

\[ \tilde{\xi}(r, 0) = \tilde{B}_1(r, 0) = \tilde{\rho}_1(r, 0) = \tilde{p}_1(r, t) = 0 \]

\[ \partial \tilde{\xi}(r, t)/\partial t \equiv \tilde{V}_1(r, 0) \neq 0 \]

• These conditions correspond to the situation when at \( t = 0 \) the plasma is in its exact equilibrium but is moving away with a small velocity \( \tilde{V}_1(r, 0) \)
Initial Value Formulation: Linearized Equations

**Mass:** \[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \quad \Rightarrow \quad \frac{\partial \tilde{\rho}_1}{\partial t} + \nabla \cdot \rho_0 \tilde{\mathbf{V}}_1 = 0 \quad \Rightarrow \quad \tilde{\rho}_1 = -\nabla \cdot \rho_0 \tilde{\xi} \]

**Energy:** \[ \frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{V} = 0 \quad \Rightarrow \quad \frac{\partial \tilde{p}_1}{\partial t} + \tilde{\mathbf{V}}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \tilde{\mathbf{V}}_1 = 0 \quad \Rightarrow \quad \tilde{p}_1 = -\tilde{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \tilde{\xi} \]

**Faraday’s law:** \[ \frac{\partial B}{\partial t} = \nabla \times (\mathbf{V} \times B) \quad \Rightarrow \quad \frac{\partial \tilde{B}_1}{\partial t} = \nabla \times (\tilde{\mathbf{V}}_1 \times B_0) \quad \Rightarrow \quad \tilde{B}_1 = \nabla \times (\tilde{\xi} \times B_0) \]

**Ampere’s law:** \[ (4\pi/c) \mathbf{J} = \nabla \times \mathbf{B} \quad \Rightarrow \quad (4\pi/c) \tilde{\mathbf{J}}_1 = \nabla \times \tilde{\mathbf{B}}_1 \quad \Rightarrow \quad (4\pi/c) \tilde{\mathbf{J}}_1 = \nabla \times \nabla \times (\tilde{\xi} \times B_0) \]

**∇ · B:** \[ \nabla \cdot B = 0 \quad \Rightarrow \quad \nabla \cdot \tilde{\mathbf{B}}_1 = 0 \quad \Rightarrow \quad \text{redundant} \]

**Mom-m:** \[ \rho \frac{d\mathbf{V}}{dt} = \frac{c^{-1}}{} \mathbf{J} \times B - \nabla p \quad \Rightarrow \quad \rho_0 \frac{d\tilde{\mathbf{V}}_1}{dt} = c^{-1} \tilde{\mathbf{J}}_1 \times B_0 \quad \Rightarrow \quad \rho_0 \frac{\partial^2 \tilde{\xi}}{\partial t^2} = \mathbf{F}(\tilde{\xi}) \]
Initial Value Formulation: Summary 1

\[ \rho_0 \frac{\partial^2 \tilde{\xi}}{\partial t^2} = F(\tilde{\xi}), \]

\[ F(\tilde{\xi}) = \frac{1}{4\pi} (\nabla \times \tilde{Q}) \times B_0 + \frac{1}{4\pi} B_0 \times (\nabla \times \tilde{Q}) + \nabla (\tilde{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \tilde{\xi}), \quad \tilde{Q} \equiv \nabla \times (\tilde{\xi} \times B_0) = \tilde{B}_1 \]

\[ + \]

\[ \tilde{\xi}(r, 0) = 0, \quad \frac{\partial \tilde{\xi}(r, 0)}{\partial t} = \text{given} \]

\[ + \]

boundary conditions
Initial Value Formulation: Summary II

• Advantages
  – directly gives time evolution of the system
  – fastest growing mode automatically appears
  – good start for a nonlinear calculation

• Disadvantages
  – more information is contained than is required for determining stability
  – extra analytical and numerical work is required to obtain this information
Boundary Conditions I

- Plasma is surrounded by a perfectly conducting wall ($n$ is outward-pointing normal vector)

- $(n \times E)|_{r_{wall}} = 0$

- $(n \cdot B)|_{r_{wall}} = 0$

- $n \times (E + c^{-1}V \times B)|_{r_{wall}} = 0 \Rightarrow (n \cdot V)|_{r_{wall}} = 0 \Rightarrow (n \cdot \tilde{\xi})|_{r_{wall}} = 0$
Boundary Conditions II

- Plasma is surrounded by a **vacuum region** (which is described by $\nabla \times \hat{B} = \nabla \cdot \hat{B} = 0$, with $\hat{B}$ the vacuum magnetic field)
  - $(n \cdot \hat{B})|_{r_{wall}} = 0$
  - plasma surface is free to move $\Rightarrow (n \cdot \tilde{\xi})|_{r_{plasma}} = \text{arbitrary}$
  - $[[n \cdot B]]|_{r_{plasma}} = 0$ (with $[[\cdots]]$ denoting a jump across the plasma surface)
  - $[[n \times B]]|_{r_{plasma}} = (4\pi/c)K$, with $K$ the surface current density
  - $[[p + B^2/8\pi]]|_{r_{plasma}} = 0$

- See Sec. 3.2 of the *Ideal Magnetohydrodynamics* for further discussion
Normal Mode Formulation I

- Assume for perturbed quantities $\tilde{q}_1(r, t) = q_1(r) \exp(-i\omega t)$

- Obtain

$$ -\omega^2 \rho_0 \xi = F(\xi), $$

$$ F(\xi) = \frac{1}{4\pi} (\nabla \times Q) \times B_0 + \frac{1}{4\pi} B_0 \times (\nabla \times Q) + \nabla (\xi \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \xi), \quad Q \equiv \nabla \times (\xi \times B_0) = B_1 $$

+ boundary conditions

- $\partial \tilde{\xi}(r, 0)/\partial t$ can be decomposed into normal modes, and each normal mode can be analyzed separately

- From now on will drop for simplicity all subscripts "0" denoting equilibrium quantities
Normal Mode Formulation II

• Advantages
  – the method is more amenable to analysis
  – directly addresses stability question (sign of $\text{Im}(\omega)$)
  – more computationally efficient

• Disadvantages
  – cannot be generalized for nonlinear calculations
  – still relatively complicated

• The usefulness of this approach relies upon the assumption that for the problems of interest the eigenvalues are discreet and distinguishable so that the concept of exponential stability is valid
Properties of the Force Operator $F(\xi)$

- To proceed further (variational approach, Energy Principle) we need to understand the properties of the force operator $F(\xi)$

- We will next see that
  - $F(\xi)$ is self-adjoint
  - $\omega^2$ is purely real (so that $\omega$ is either purely real or purely imaginary)
  - the normal modes are orthogonal
Self-adjointness of $F(\xi)$

- It is **associated** with the **conservation of energy** (there is no dissipation in the system)

- $F$ is self-adjoint if for any arbitrary, independent vectors $\xi$ and $\eta$ satisfying appropriate boundary conditions

$$\int dr \; \eta \cdot F(\xi) = \int dr \; \xi \cdot F(\eta)$$

- Examples of **self-adjoint** operators: $\int dr \; \eta \cdot \xi$, $\int dr \; \eta \cdot \frac{\partial^2 \xi}{\partial x^2} = \int dr \; \xi \cdot \frac{\partial^2 \eta}{\partial x^2}$

- Example of a **non-self-adjoint** operator: $\int dr \; \eta \cdot \frac{\partial \xi}{\partial x} = - \int dr \; \xi \cdot \frac{\partial \eta}{\partial x}$
Self-adjointness of $F(\xi)$

- Can demonstrate self-adjointness directly via a tedious calculation. E.g. assuming $n \cdot \xi = n \cdot \eta = 0$ on plasma boundary can show

$$\int dr \, \eta \cdot F(\xi) = - \int dr \left[ \frac{(B \cdot \nabla \xi_{\perp})(B \cdot \nabla \eta_{\perp})}{4\pi} + \gamma p(\nabla \cdot \xi)(\nabla \cdot \eta) ight.$$

$$+ \frac{B^2}{4\pi} (\nabla \cdot \xi_{\perp} + 2\kappa \cdot \xi_{\perp})(\nabla \cdot \eta_{\perp} + 2\kappa \cdot \eta_{\perp})$$

$$- \frac{B^2}{\pi} (\kappa \cdot \xi_{\perp})(\kappa \cdot \eta_{\perp}) + (\xi_{\perp} \eta_{\perp} : \nabla \nabla) \left(p + \frac{B^2}{8\pi}\right) \right].$$

- In obtaining this expression wrote $\xi = \xi_{\perp} + \xi_{\parallel} b$ and $\eta = \eta_{\perp} + \eta_{\parallel} b$, where $\parallel$ and $\perp$ refer to the equilibrium $B$

- $\kappa \equiv b \cdot \nabla b$ is the magnetic field curvature

- Can generalize the proof for the boundary conditions allowing a vacuum region
Real $\omega^2$

- Dot the equation $-\omega^2 \rho \xi = F(\xi)$ with $\xi^*$ and integrate over the plasma volume:

$$\omega^2 \int dr \rho |\xi|^2 = - \int dr \xi^* \cdot F(\xi)$$

- Repeat for the complex conjugate equation $-(\omega^*)^2 \rho \xi^* = F(\xi^*)$ dotted with $\xi$:

$$(\omega^*)^2 \int dr \rho |\xi|^2 = - \int dr \xi \cdot F(\xi^*)$$

- Because of self-adjointness of $F$,

$$[\omega^2 - (\omega^*)^2] \int dr \rho |\xi|^2 = 0 \implies \omega^2 = (\omega^*)^2$$

- So, both $\omega^2$ and $\xi$ are purely real

- Consequence: $\omega^2 > 0 \implies$ stable oscillations, $\omega^2 < 0 \implies$ exponential instability
Orthogonality of the Normal Modes

- Consider two discreet normal modes \((\xi_m, \omega^2_m)\) and \((\xi_n, \omega^2_n)\) for which
  \[
  -\omega^2_m \rho \xi_m = F(\xi_m), \quad -\omega^2_n \rho \xi_n = F(\xi_n)
  \]

- Dot the \(\xi_m\)-equation with \(\xi_n\) and vice versa, integrate over the plasma volume, subtract the results, and use self-adjointness of \(F\) to get
  \[
  (\omega^2_m - \omega^2_n) \int d\mathbf{r} \rho \xi_m \cdot \xi_n = 0
  \]

- For non-degenerate discreet normal modes \(\omega^2_m \neq \omega^2_n\) so that
  \[
  \int d\mathbf{r} \rho \xi_m \cdot \xi_n = 0
  \]

- Therefore, such modes are orthogonal with weight function \(\rho\)
Elements of Variational Calculus I

- Consider the classic Sturm-Liouville problem [$f(x)$ and $g(x)$ are known]

\[
\frac{d}{dx} \left( f(x) \frac{dy}{dx} \right) + (\lambda - g(x))y = 0, \quad y(0) = y(1) = 0
\]

- Need to accurately evaluate the eigenvalue $\lambda$

- Cast the differential equation in the form of an equivalent integral relation

- Substitute analytic “guesses” (trial functions) in place of the true eigenfunction to estimate the eigenvalue

- The integral relation is variational if the estimated $\lambda$ exhibits an extremum for the actual eigenfunction. This extremum is equal to the actual eigenvalue

- If the integral relation is variational the estimated eigenvalue is more accurate than the trial function
Elements of Variational Calculus II

- Multiply the Sturm-Liouville differential equation by \( y \) and integrate over \( 0 \leq x \leq 1 \)

- Employing integration by parts and boundary conditions for \( y \) obtain

\[
\lambda = \frac{\int dx \ (f y' + gy)^2}{\int dx \ y^2}
\]

- **Prove that this form is variational.** Assume that trial function \( y_0(x) \) gives \( \lambda_0 \) and trial function \( y(x) = y_0(x) + \delta y(x) \), with \( \delta y \) a small arbitrary perturbation such that \( \delta y(0) = \delta y(1) = 0 \), gives \( \lambda = \lambda_0 + \delta \lambda \). Then,

\[
\delta \lambda = \frac{\int dx \ [f (y_0 + \delta y)' + g (y_0 + \delta y)]^2}{\int dx \ (y_0 + \delta y)^2} - \frac{\int dx \ (f y_0' + g y_0)^2}{\int dx \ y_0^2}
\]
• Taylor expanding for small $\delta y$ obtain

$$\delta \lambda \approx \frac{2 \int dx \left[ \delta y' f' y_0' + \delta y y_0 (g - \lambda_0) \right]}{\int dx y_0^2}$$

• Integrating the first term by parts yields

$$\delta \lambda = -\frac{2 \int dx \delta y \left[ (f y_0')' + (g - \lambda_0)y_0 \right]}{\int dx y_0^2}$$

• If $y_0$ is a solution to the original differential equation, $\delta \lambda = 0$ for an arbitrary $\delta y$
Elements of Variational Calculus IV

- Can come up with an **infinite number of integral relations** for $\lambda$, which are **not variational**

- E.g. multiplying the original differential equation by $h(x)y(x)$ instead of $y(x)$ and integrating over $x$ gives

\[
\lambda = \frac{\int dx \left\{ hfy'^2 + \left[ hg - 0.5(fh)' \right]y^2 \right\}}{\int dx \; hy^2}
\]

- This integral relation is **not variational** since the function that makes $\delta \lambda = 0$ does not satisfy the original differential equation unless $h(x) = 1$
• Since $\delta \lambda = 0$ for the true eigenfunction $y_0(x)$, a variational estimate for $\lambda$ obtained with some trial function is more accurate than the trial function itself.

• To prove this write $y = y_0 + \delta y$, with $|\delta y|/|y_0| \ll 1$.

• Substituting $y$ into the variational expression for $\lambda$ gives

$$
\lambda = \lambda_0 + \frac{\int dx \left[ f(\delta y')^2 + g(\delta y)^2 \right]}{\int dx \ y_0^2} + O[(\delta y)^3]
$$

• The estimate $\lambda$ differs from the true eigenvalue $\lambda_0$ by a correction of order $O[(\delta y)^2]$, which is one order more accurate than the trial function.
Variational Formulation of Linearized Ideal MHD Stability Problem I

- Because of self-adjointness of $F(\xi)$ the normal mode formulation of the linearized ideal MHD stability problem can be easily cast in the variational form (Bernstein et al., 1958)

- Dotting $-\omega^2 \rho \xi = F(\xi)$ with $\xi^*$ and integrating over the plasma volume yields

$$\omega^2 = \frac{\delta W(\xi^*, \xi)}{K(\xi^*, \xi)}, \quad \delta W(\xi^*, \xi) \equiv -\frac{1}{2} \int dr \, \xi^* \cdot F(\xi), \quad K(\xi^*, \xi) \equiv \frac{1}{2} \int dr \, \rho |\xi|^2$$

- $\delta W$ represents the change in potential energy associated with the perturbation and is equal to work done against the force $F(\xi)$ in displacing the plasma by $\xi$. $K$ is proportional to the kinetic energy

- The above integral representation is variational

- To prove this let $\xi \rightarrow \xi + \delta \xi$, evaluate $\delta \omega^2$, and set it to zero
Variational Formulation of Linearized Ideal MHD Stability Problem II

\[ \omega^2 + \delta \omega^2 = \frac{\delta W(\xi^*, \xi) + \delta W(\delta \xi^*, \xi) + \delta W(\xi^*, \delta \xi) + \delta W(\delta \xi^*, \delta \xi)}{K(\xi^*, \xi) + K(\delta \xi^*, \xi) + K(\xi^*, \delta \xi) + K(\delta \xi^*, \delta \xi)} \]

- For small \( \delta \xi \),

\[ \delta \omega^2 = \frac{\delta W(\delta \xi^*, \xi) + \delta W(\xi^*, \delta \xi) - \omega^2[K(\delta \xi^*, \xi) + K(\xi^*, \delta \xi)]}{K(\xi^*, \xi)} \]

- Using self-adjointness of \( F \) and requiring \( \delta \omega^2 = 0 \) gives

\[ \int \text{d}r \left\{ \delta \xi^* \cdot [F(\xi) + \omega^2 \rho \xi] + \delta \xi \cdot [F(\xi^*) + \omega^2 \rho \xi^*] \right\} = 0 \]

- Since \( \delta \xi \) is arbitrary this implies \(- \omega^2 \rho \xi = F(\xi) \Rightarrow \text{the formulation is variational}\)
Variational Formulation of Linearized Ideal MHD Stability Problem III

- The variational formulation and the normal mode eigenvalue formulation of the linearized ideal MHD stability problem are equivalent.

- Advantages of the variational formulation
  - allows use of trial functions to estimate $\omega^2$
  - can be applied to multidimensional systems efficiently

- Disadvantages of the variational formulation
  - still somewhat complicated
  - gives more information than minimum required
Energy Principle: Preliminary Comments

- Growth times of ideal MHD instabilities are short, $\tau_{\text{inst}} \sim \frac{r}{V_{\text{thermal},i}} \lesssim 50 \ \mu s$
- Experimental times are much longer, $\tau_{\text{exp}} > 1 \ ms$
- Ideal MHD instabilities normally terminate the plasma
- Therefore, it is usually much more important to know whether the system is ideal MHD stable or not and to determine the conditions for avoiding instabilities, rather than to know precise growth rates (which can easily be estimated)
- In these cases, the variational formulation can be simplified further, giving the Energy Principle
- The Energy Principle exactly determines stability boundaries but only estimates growth rates
Energy Principle: Formulation

- Variational formulation:

\[ \omega^2 = \frac{\delta W}{K}, \quad \delta W = -\frac{1}{2} \int dr \, \xi^* \cdot F(\xi), \quad K = \frac{1}{2} \int dr \, \rho |\xi|^2 \]

If all \( \omega^2 \geq 0 \), then the system is **stable**
If any \( \omega^2 < 0 \), then the system is **unstable**

- Energy Principle:

If \( \delta W(\xi^*, \xi) \geq 0 \) for all allowable \( \xi \), then the system is **stable**
If \( \delta W(\xi^*, \xi) < 0 \) for any \( \xi \), then the system is **unstable**

- The underlying physics: a system is unstable if a perturbation can reduce its potential energy
Energy Principle: Proof

• Assume that $F$ allows only discreet normal modes, $\xi_n$, which form a complete set of basis functions (see the textbook for a more general proof):

$$-\omega_n^2 \rho \xi_n = F(\xi_n), \quad \int dr \rho \xi_n^* \cdot \xi_m = \delta_{nm}$$

• Can expand an arbitrary trial function as $\xi(r) = \sum_n a_n \xi_n(r)$

• Evaluate $\delta W$:

$$\delta W = -\frac{1}{2} \int dr \xi^* \cdot F(\xi) = -\frac{1}{2} \sum_{n,m} a_n^* a_m \int dr \xi_n^* \cdot F(\xi_m)$$

$$= -\frac{1}{2} \sum_{n,m} a_n^* a_m \int dr \xi_n^* \cdot (-\omega_m^2 \rho \xi_m) = \frac{1}{2} \sum_m \omega_m^2 |a_m|^2$$

• It a $\xi$ exists that makes $\delta W < 0 \Rightarrow$ at least one $\omega_m^2 < 0 \Rightarrow$ instability

• If all $\xi$ make $\delta W > 0 \Rightarrow$ all $\omega_m^2 > 0 \Rightarrow$ stability
Extended Energy Principle I

- The Energy Principle calculated with

\[ F(\xi) = \frac{1}{4\pi}(\nabla \times Q) \times B_0 + \frac{1}{4\pi}B_0 \times (\nabla \times Q) + \nabla (\xi \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \xi), \quad Q \equiv \nabla \times (\xi \times B_0) = B_1 \]

is valid both when the plasma is surrounded by a conducting wall and when it is isolated from this wall by a vacuum region.

- In the first case application of the Energy Principle is straightforward since the only boundary condition for \( \xi \) is \( (n \cdot \xi_\perp)|_{r_{wall}} = 0 \).

- When the vacuum region is present, direct application of this form of the Energy Principle is cumbersome because
  - boundary conditions for \( \xi \) on the plasma surface are complicated
  - this form of \( \delta W \) is not very intuitive since the vacuum contribution does not appear explicitly.
Extended Energy Principle II

• After a lengthy calculation can rewrite $\delta W$ in the following intuitive form

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

- fluid energy
- surface energy
- vacuum energy

• The surface energy, $\delta W_S = \frac{1}{2} \int_S dS |n \cdot \xi_\perp|^2 n \cdot \left[ \nabla \left( p + \frac{B_1^2}{8\pi} \right) \right]$

• $[[q]]$ denotes a jump in $q$ from vacuum to plasma; $\delta W_S = 0$ unless plasma surface currents are present

• The vacuum energy, $\delta W_V = \frac{1}{2} \int_V d\mathbf{r} \frac{\hat{B}_1|^2}{4\pi}$

• Vacuum magnetic field is obtained from $\nabla \times \hat{B}_1 = \nabla \cdot \hat{B}_1 = 0$, subject to the boundary conditions

$$\left. (n \cdot \hat{B}_1) \right|_r_{wall} = 0, \quad \left. (n \cdot \hat{B}_1) \right|_r_{plasma} = \left. \hat{B}_1 \cdot \nabla (n \cdot \xi_\perp) - (n \cdot \xi_\perp)(n \cdot \nabla \hat{B}_1 \cdot n) \right|_r_{plasma}$$
Extended Energy Principle III

- The fluid energy,

\[ \delta W_F = \frac{1}{2} \int_P \text{d}r \left[ \frac{|Q_\perp|^2}{4\pi} + \frac{B^2}{4\pi} |\nabla \cdot \xi_\perp + 2\kappa \cdot \xi_\perp|^2 + \gamma p |\nabla \cdot \xi|^2 \right] - 2(\nabla p \cdot \xi_\perp)(\kappa \cdot \xi^*_\perp) - J_\parallel (\xi^*_\perp \times b \cdot Q_\perp) \]

- Individual terms have the following physical interpretation:
  - \(|Q_\perp|^2\) represents magnetic field bending energy \(\Rightarrow\) stabilizing
  - \(|\nabla \cdot \xi_\perp + 2\kappa \cdot \xi_\perp|^2\) represents magnetic field compression energy \(\Rightarrow\) stabilizing
  - \(\gamma p |\nabla \cdot \xi|^2\) represents plasma compression energy \(\Rightarrow\) stabilizing
  - \(\nabla p\) represents pressure gradient instability drive \(\Rightarrow\) can be destabilizing
  - \(J_\parallel\) represents parallel current instability drive \(\Rightarrow\) can be destabilizing
Incompressibility I

- $\xi_\parallel$ only contributes to $\delta W$ through the term $\delta W_\parallel \equiv \int_P d\mathbf{r} \gamma p |\nabla \cdot \xi|^2$

- Let’s minimize $\delta W_\parallel$ with respect to $\xi_\parallel$ by allowing $\xi_\parallel \to \xi_\parallel + \delta \xi_\parallel$ and then setting the variation of $\delta W_\parallel$ to zero:

$$\delta (\delta W_\parallel) = \int_P d\mathbf{r} \gamma p (\nabla \cdot \xi) \nabla \cdot \left( \frac{\delta \xi_\parallel B}{B} \right) = - \int_P d\mathbf{r} \frac{\delta \xi_\parallel}{B} B \cdot \nabla [\gamma p (\nabla \cdot \xi)]$$

$$= - \int_P d\mathbf{r} \frac{\delta \xi_\parallel}{B} \gamma p B \cdot \nabla (\nabla \cdot \xi)$$

- The boundary term vanishes since $\mathbf{n} \cdot B = 0$ on the plasma surface. We also used the fact that $B \cdot \nabla p = 0$

- The minimizing condition is $B \cdot \nabla (\nabla \cdot \xi) = 0$
Incompressibility II

- For most configurations the operator $B \cdot \nabla$ is \textbf{nonsingular} and the minimizing condition becomes $\nabla \cdot \xi = 0$.

- The \textbf{most unstable modes are incompressible} $\Rightarrow \delta W_\parallel = \int_P \gamma p |\nabla \cdot \xi|^2 = 0$.

- There are \textbf{two cases} when $\nabla \cdot \xi$ cannot be set to zero:
  - system possesses a special symmetry ($B \cdot \nabla$ vanishes identically for certain specific modes)
  - closed field line systems ($B \cdot \nabla$ is not singular but a periodicity constraint must be taken into account)
Special Symmetry: $m = 0$ mode in Z-pinch

- $B = B_\theta(r)\hat{\theta}, \quad \xi = \xi(r) \exp[i(m\theta + kz)]$

\[
\nabla \cdot \xi = \nabla \cdot \xi_\perp + B \cdot \nabla \left( \frac{\xi_\parallel}{B} \right) = \nabla \cdot \xi_\perp + \frac{B_\theta}{r} \frac{\partial}{\partial \theta} \left( \frac{\xi_\parallel}{B_\theta} \right) = \nabla \cdot \xi_\perp + \frac{im\xi_\parallel}{r}
\]

- For $m = 0$ $\xi_\parallel$ does not appear, so that $\nabla \cdot \xi = \nabla \cdot \xi_\perp$ and

\[
\delta W_\parallel = \int_P \text{d}r \, \gamma p \left| \nabla \cdot \xi_\perp \right|^2
\]
Close Field Line Systems

- Have to satisfy a **periodicity constraint on each field line**, $\xi_{\|}(l) = \xi_{\|}(l + L)$, where $l$ is arc length and $L$ is the total length of a given line.

- Need to add a homogeneous solution when solve $B \cdot \nabla (\nabla \cdot \xi) = 0$:

  \[
  \nabla \cdot \xi = F(p) \quad \Rightarrow \quad \frac{\xi_{\|}}{B} = - \int_0^l \frac{dl}{B} \nabla \cdot \xi_\perp + F(p) \int_0^l \frac{dl}{B}
  \]

- Then, periodicity requires

  \[
  F(p) = \langle \nabla \cdot \xi_\perp \rangle \equiv \oint \frac{dl}{B} \nabla \cdot \xi_\perp
  \]

- Consequently,

  \[
  \delta W_{\|} = \frac{1}{2} \int_P d\mathbf{r} \gamma p |\langle \nabla \cdot \xi_\perp \rangle|^2
  \]
Classification of MHD Instabilities

- **Internal or fixed boundary modes.** Plasma surface is not perturbed, \((n \cdot \xi_\perp)_{r_{\text{plasma}}} = 0\). Equivalent to moving conducting wall onto the plasma surface.

- **External or free boundary modes.** Plasma surface is allowed to move, \((n \cdot \xi_\perp)_{r_{\text{plasma}}} \neq 0\). Often the most severe stability limitations.

- **Current driven (or kink) modes.** \(J_\parallel\) is the dominant destabilizing term. Important in tokamaks, RFPs (Kruskal-Shafranov instability, sawtooth oscillations, disruptions). In general, the modes have long wavelengths, low \(m, n\). Cure: tight aspect ratio, low current, peaked current profiles, conducting wall.

- **Pressure driven modes.** \(\nabla p\) is the dominant destabilizing term. Special cases: interchange or flute modes, ballooning modes, sausage instability. Important in tokamaks, stellarators, RFPs, mirrors. Normally have long parallel and short perpendicular wavelengths. Cure: low \(\beta\), magnetic field shear, average favorable curvature.
Summary I

- We reviewed basics of ideal MHD stability theory

- A plasma is said to be exponentially stable if $\text{Im}(\omega) \leq 0$ for every discreet normal mode of the system

- General 3D stability problem is first formulated as an initial value problem and then transformed into a normal mode eigenvalue problem: $-\omega^2 \rho \xi = F(\xi)$

- The force operator $F$ is self-adjoint, so that $\omega^2$ is purely real. $\omega^2 > 0$ indicates stability, $\omega^2 < 0$ – instability

- Using self-adjointness of $F$ can recast the normal mode eigenvalue problem into the form of a variational principle

- If only stability boundaries are required, can obtain from the variational principle a powerful minimizing principle, known as the Energy Principle. The system is stable if and only if $\delta W \geq 0$ for all allowable plasma displacements
Summary II

- Can minimize $\delta W$ once and for all with respect to $\xi_\parallel$ for arbitrary geometries. For most systems this leads to $\nabla \cdot \xi = 0$, i.e. the most unstable perturbations are incompressible.

- Based on this analysis, ideal MHD instabilities can be conveniently separated into internal (fixed boundary) or external (free boundary), and current driven (kinks) of pressure driven.
Example: Stability of $\theta$ Pinch I

- $B = B_z(r)\hat{z}, \ p = p(r)$

Equilibrium
- $\nabla \cdot B = 0$ is satisfied identically
- Ampere’s law gives $J = J_\theta(r)\hat{\theta} = -(c/4\pi)dB_z(r)/dr$
- pressure balance gives $d(p + B_z^2/8\pi)/dr = 0$

$$p(r) + \frac{B_z(r)^2}{8\pi} = \frac{B_0^2}{8\pi}$$

- $B_0$ is the externally applied field

Equilibrium is independent of $\theta$ and $z \Rightarrow$ can write plasma displacement as
$$\xi(r) = \xi(r)\exp[i(m\theta + k z)], \text{ with } m \text{ and } k \text{ poloidal and axial wave numbers}$$

$\nabla \cdot \xi = 0 \Rightarrow \nabla_\parallel \xi_z = (i/kr)[(r\xi_r)' + im\xi_\theta] \Rightarrow$ as long as $k \neq 0$ the most unstable perturbations are incompressible
Example: Stability of $\theta$ Pinch II

- Next, evaluate $\delta W_F$

- It is easy to show that

  - $Q_\perp = ik B_z \xi_\perp = ik B_z (\xi_r \hat{r} + \xi_\theta \hat{\theta}) \Rightarrow |Q_\perp|^2 = k^2 B_z^2 (|\xi_r|^2 + |\xi_\theta|^2)$

  - $\nabla \cdot \xi_\perp = \frac{1}{r} (r \xi_r)' + \frac{im}{r} \xi_\theta \Rightarrow$
    \[ |\nabla \cdot \xi_\perp|^2 = \frac{1}{r^2} (r \xi_r)'|^2 + \frac{m^2}{r^2} |\xi_\theta|^2 + \frac{im}{r^2} [(r \xi_r^*)' \xi_\theta - (r \xi_r)' \xi_\theta^*] \]

- $\kappa = 0$

- $J_\parallel = 0$

- Then ($Z$ is the length of the plasma column, $a$ is its radius),

  \[ \delta W_F = \frac{Z}{4} \int_0^a dr \ W(r)r, \]

  \[ W(r) = B_z^2 \left\{ k^2 (|\xi_r|^2 + |\xi_\theta|^2) + \frac{|(r \xi_r)'|^2 + m^2 |\xi_\theta|^2 + im[(r \xi_r^*)' \xi_\theta - (r \xi_r)' \xi_\theta^*]}{r^2} \right\} \]
Example: Stability of \( \theta \) Pinch III

- \( \xi_\theta \) appears only algebraically

- use identity \( |a \xi_\theta + b|^2 = a^2|\xi_\theta|^2 + a(b^* \xi_\theta + b \xi_\theta^*) + |b|^2 \), valid for arbitrary real \( a \) and complex \( b \) and \( \xi_\theta \), to combine all \( \xi_\theta \) terms by completing the squares as follows (\( k_0^2 \equiv k^2 + m^2/r^2 \)):

\[
W(r) = B_z^2 \left\{ \left| k_0 \xi_\theta - \frac{im}{k_0 r^2} (r \xi_r)' \right|^2 + \frac{k^2}{k_0^2 r^2} \left[ |(r \xi_r)'|^2 + k_0^2 r^2 |\xi_r|^2 \right] \right\}
\]

- \( \xi_\theta \) appears only in the first term. Since this term is positive, its minimum value is zero. So, \( \delta W_F \) is minimized by choosing

\[
\xi_\theta = \frac{im}{k^2 r^2 + m^2} (r \xi_r)'
\]
Example: Stability of $\theta$ Pinch IV

- Then,

$$\delta W_F = \frac{Z}{4} \int_0^a dr \ r \ \frac{k^2 B_z^2}{k^2 r^2 + m^2} \left[ |(r\xi'_r)|^2 + (k^2 r^2 + m^2)|\xi'_r|^2 \right]$$

- $\delta W_F > 0$ for any $k^2 > 0$ and $\delta W_F \to 0$ as $k^2 \to 0$

- So, for any $\beta$, $\theta$ pinch is positively stable for perturbations with finite axial wavelengths and marginally stable for perturbations with very long axial wavelengths

- Inclusion of the vacuum term removes the restriction that $\xi_r(a) = 0$ but does not alter the stability: $\delta W_F$ remains positive, $\delta W_V \geq 0$ by definition, and $\delta W_S = 0$ for profiles that decay smoothly to zero at $r = a$
Example: Stability of $\theta$ Pinch V

- Physical explanation for the stability:
  - $J_\parallel = 0 \Rightarrow$ no current driven modes
  - $\kappa = 0 \Rightarrow$ no pressure driven modes
  - any perturbation bends or compresses $B \Rightarrow$ both are stabilizing influences
Homework: Stability of $Z$ Pinch

- Use Ampere’s law and pressure balance equation to obtain a $Z$ pinch equilibrium condition

- Use energy principle to analyze stability of this equilibrium
  - write the displacement as $\xi(r) = \xi(r) \exp[i(m\theta + kz)]$
  - consider cases of $m = 0$ and $m \neq 0$ modes separately
  - show that $m \neq 0$ are incompressible, while compressibility must be retained for $m = 0$ modes
  - in each case, write $\delta W_F$ and show that $\xi_z$ enters only algebraically
  - complete squares with respect to $\xi_z$ and obtain expressions for $\xi_z$ that minimize $\delta W_F$
  - observe, that for the $m \neq 0$ case the most unstable modes have $k \to \infty$, and take this limit
  - obtain necessary and sufficient stability conditions for $m \neq 0$ modes $[8\pi rp' + m^2B_\theta^2 > 0]$ and $m = 0$ modes $[-rp'/p < 2\gamma B_\theta^2/(4\gamma p + B_\theta^2)]$
  - come up with a physical picture for the instability