A Simple Theory of the 2D Child-Langmuir Law

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First Analytic Theory of 2D Child-Langmuir Law

2D Child-Langmuir Law is of great interest:

- Electron emission from finite areas
- Sightings of higher-than-$J_{CL}(1D)$ emission on novel cathodes
- Edge emission on e-guns and high power diodes
- Extremely difficult to solve 2D force law, Poisson eq., and continuity eq. analytically
Analytic Theory of 2D Child-Langmuir Law (cont.)

The 2D Child-Langmuir Law was derived analytically from First Principles

- The theory is extremely simple
- It is in excellent agreement with OOPIC and MAGIC simulations
- Scaling laws for a 3D geometry are proposed
2D Child-Langmuir Law

\[ J(1D) = \frac{4G_0}{9} \sqrt{\frac{2e}{m}} \frac{V^{3/2}}{D^2} \]

From PIC code data,

\[ \frac{J(2D)}{J(1D)} \cong 1 + 0.3145 \left( \frac{1}{W/D} \right), \quad W/D > 0.1 \]

That universal function is exceedingly difficult to derive analytically. [See Eq. (2) below.]

The simulation shows the following scenario for the breakdown of the laminar flow as $J \rightarrow J_{CL}(2)$. The electric field in the vicinity of the center of the emitting strip to within 5%, for all values of magnetic field: $B = 0$, 0.01, and 100 T.

$$\frac{J_{CL}(2)}{J_{CL}(1)} = \left[ 1 + \frac{0.3145}{W/D} - \frac{0.0004}{(W/D)^2} \right]$$ (2)
\[
E(x = 0, z = 0) = \int_0^D dz \int_{-\frac{W}{2}}^{\frac{W}{2}} dx \left[ \frac{\rho(x, z)}{4\pi \varepsilon_0 \sqrt{x^2 + z^2}} \right] \cdot \cos \theta \frac{z}{\sqrt{x^2 + z^2}}
\]

\[
\approx \int_0^D dz \frac{\rho(z)}{2\varepsilon_0} \tan^{-1} \left( \frac{W}{2z} \right) \approx \int_0^D dz \frac{\rho(z)}{2\varepsilon_0} \left[ \frac{\pi}{2} - 2z \frac{2}{W} \right]
\]

\[
G \ J(2D) \int_0^D dz \frac{1}{2\varepsilon_0 v} \left[ \frac{\pi}{2} - 2z \frac{2}{W} \right] = V/D
\]

\[
G \ J(1D) \int_0^D dz \frac{1}{2\varepsilon_0 v} \left[ \frac{\pi}{2} \right] = V/D
\]
\[
\frac{J(2D)}{J(1D)} \approx 1 + \frac{1}{\pi} \frac{D}{W} \left[ \int_0^{D} \frac{dz}{v(z)} \right] \int_0^{D} \frac{dz}{v(z)}
\]

Let \( v(z) = Cz^{2/3} \)

\[
\Rightarrow \frac{J(2D)}{J(1D)} \approx 1 + \frac{1}{\pi} \frac{D}{W} \left[ 0.3183 \right]
\]

\[
\frac{J(2D)}{J(1D)} \approx 1 + 0.3145 \frac{D}{W}
\]

[\( e\phi(z) = 1/2mv^2 \propto z^{3/4} \]

from 1D C-L solution]

[From OOPIC, MAGIC]
Scaling for Higher-dimensional Child-Langmuir Law

\[
\frac{J(2)}{J(1)} \cong 1 + \frac{1}{\pi} \frac{D}{W}
\]

\[
\frac{J(2)}{J(1)} \cong 1 + \frac{1}{4} \frac{D}{R}
\]

\[
\frac{J(3)}{J(1)} \cong 1 + \frac{1}{\pi} \frac{D}{W} + \left( \frac{1}{4} - \frac{1}{2\pi} \right) \frac{D}{R}
\]